Knotted Ribbons and Ribbon Length

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Abstract

This paper will describe a three-dimensional model used to generate knotted ribbons with minimal ribbon length, the ratio of the length of a ribbon to its width. This model treats ribbons as modified ropes and constructs them in three-dimensional space. It is shown that the ribbon length for all nontrivial framed unknots is 1, and that a lower bound for the ribbon length of all ribbons with knotted cores is also 1. This will be compared to other ribbon length conjectures already published.

1 Introduction

Knots have frequently been studied as both physical and abstract mathematical objects. This paper will treat knots as ribbons in three-dimensional space. A ribbon can be thought of as a subset of $\Re^3$ homeomorphic to a annulus [4]. Knotting the *core*, or centerline, of the ribbon is analogous to tying an actual knot in the ribbon, pulling it tight until it becomes a flat folded knotted strip, and then joining the ends together with a minimal amount of excess ribbon. This joining creates what is called the closed form of the knot. A closed knot can also be formed by introducing n half-twists in a ribbon, then joining the ends together. The core of ribbons constructed this way is always the unknot, and the boundary of the ribbon forms a link. The boundaries form a class of knots called Torus knots, denoted $T_{2,n}$. Adding simply one half-twist to a ribbon and joining the ends together creates the familiar Mobius strip. Torus knots are also produced by knotting the core of the ribbon itself. Figures 1 and 2 show the trefoil knot $T_{2,3}$ both as the boundary of a twisted ribbon with the unknot as core and as a ribbon with a knotted core. The core is shown as the dashed line, and a dotted line is shown to indicate that the ribbon has been joined together.

Knotted ribbons are distinguished by both type of core K and n, the linking number of the boundary with the core, and are usually referred to as (K, n). The *linking number* of the boundary of a ribbon, b with its core, K is defined as

$$n = lk(b, K) = \left(\frac{1}{2} \sum_{c} \epsilon(c) \right),$$

where c denotes the crossings *between* the boundary and the core and $\epsilon(c)$ denotes the signs of the crossings. The crossing signs on an oriented knot diagram can be defined as follows. A crossing is positive if when traveling along the

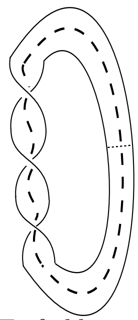


Figure 1: Trefoil knotted ribbon with unknot core

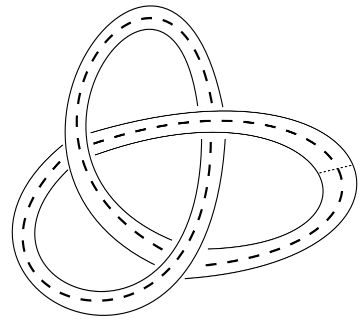


Figure 2: Ribbon with trefoil knotted core

undercrossing, the overcrossing runs from left to right. For example, in Figure 1, the linking number is $n = \pm 3$, while in Figure 2, $n = \pm 6$. $(K, 0)$ is known as a *trivial framing* of the knot K.

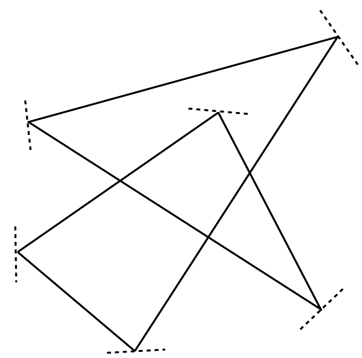
This paper will compare and contrast various notions of ribbon length.

Definition The *ribbon length* of a knotted ribbon is the ratio of the length of the core to the width of the ribbon itself. Let $\overline{K}$ be any framed knot topologically equivalent to (K, n). The ribbon length $rl(K, n)$ of a framed knot (K, n) is $rl(K, n) = \inf_{\overline{K}} rl(\overline{K})$

Ribbon length is generally denoted as $rl(K, n) = x$, where x is the infimum ribbon length for the knot. The length of a ribbon is calculated by measuring the length of its core. The ribbon length of the knot is the minimum of these over all realizations of the knot. This paper will prove a theorem that the ribbon length for all nontrivial knotted ribbons, such as Figure 1, with the unknot as the core is 1. It will also be shown that 1 is a lower bound for framed ribbons with knotted cores, such as Figure 2.

2 Kauffman's Model

Knotted ribbons have already been investigated in knot theory. In a paper by Louis Kauffman [4], a model is described for creating ribbons with knotted cores, and conjectures are offered as to the ribbon length for two common knots, the trefoil knot ($T_{2,3}$) and the figure eight knot using this model. In Kauffman's model, the core of a knotted ribbon is represented by a piecewise linear curve embedded in the plane. At each angle in the embedding, a mirror segment is placed perpendicular to the bisector of the angle, which represents a fold in the ribbon made at that angle. Over- and undercrossings are then chosen and the width of the core is expanded maximally (until the ribbon can no longer be folded) to create the ribbon. Figure 2 shows the piecewise linear core with mirrors in place.



Kauffman conjectures that this model projects ribbons with minimum ribbon length for both the trefoil and figure eight knots. His model, however, cannot account for twists in the ribbon that do not occur at a mirror segment or fold. Using his model, only single folds made in the ribbon can add twists to the ribbon. Each fold, however, increases the length required to make and close the knot, subsequently *increasing* the ribbon length. In addition, each single fold changes the orientation of the ribbon when it exits the fold. In order to reverse the orientation of the ribbon using Kauffman's model, another fold must be introduced. For example, in order to create the Mobius strip with this configuration, three mirrors must be used to allow for the orientation of the paper to change exactly once. This paper offers an alternate model for knotted ribbons that allows for twists to be made without the introduction of more mirrors.

A *knotted ribbon universe* consists of a polygon immersed in the plane together with mirrors placed at each of its vertices. These mirrors may be one of two forms:

- A *single mirror*, which will change the orientation of the ribbon once constructed;
- or a *double mirror*, which will maintain the orientation of the ribbon once it is constructed.

A *knotted ribbon* is formed by choosing over- and undercrossings at each self transversal, declaring the orientation of the fold, then expanding the width of the ribbon and threading it properly. If we are traveling along a ribbon in a certain direction, then we will define the fold made to be oriented *positively* if the part folded lies *behind* the part we are traveling on, and is oriented *negatively* if that part lies *in front of* the other.

Using this construction, a Mobius strip may be constructed with only one double mirror. This can be made with the smallest ribbon length by using square sheet of paper. A double mirror can be inserted in the center of the square, and the figure can be folded along both diagonals of the square and the ends can be joined to be created. Figure 3 shows the configuration on a square.

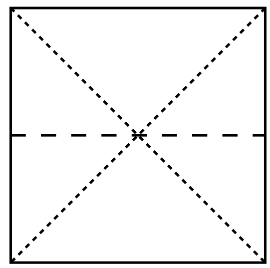


Figure 3: Double mirror configuration for Mobius strip

The dashed line denoted the core of the ribbon, while the dotted lines along the diagonals of the square denote the mirrors where folds are to be made. The result of this double mirror folding is a Mobius strip $T_{2,1}$ with a ribbon of equal width and length and thus having ribbon length 1. By connecting two of these minimal Mobius strips with the correct orientation (i.e. so they do not cancel each other out to form the unknot), a Hopf link $T_{2,2}$ can actually be created with paper which has ribbon length 2. Similarly, a trefoil can be created with 3 squares, but one extra square is needed to join the ends of the ribbon back together, thus making a paper constructable trefoil with ribbon length of 4. In general, any $T_{2,n}$ knot, $n \geq 3$, is paper constructable to have ribbon length $n + 1$. We conjecture this to be the optimal constructable ribbon length for these knotted ribbons.

3 Barr's Model

Stephen Barr [2], in his book *Experiments in Topology*, describes a model for constructing the shortest Mobius strip out of a piece of paper, comparable to a ribbon. This model describes creating a Mobius strip knot $T_{2,1}$ (a half-twisted ribbon) as turning one edge of the paper over and then joining the ends together. This equates to joining A to A' and B to B' in Figure 4. This Mobius strip begins with a piece of paper of length $1/\sqrt{3}$ (AB) and width 1 (AB'), with folds introduced creating equilateral triangles as shown in Figure 5. The first fold is over on the dotted lines AC' and $A'C$. Another fold is introduced along the dashed diagonal, AA', and line segments BC' and $B'C$ are then joined.

The figure is then folded in half along CC' to make the corners A and A' meet, and the line segments $A'C$ and AC' are joined, finishing the construction. A half-twist has been introduced to the resulting figure, though hard to believe, and thus a Mobius band with length less than its width (or ribbon length *less* than one) has been created.

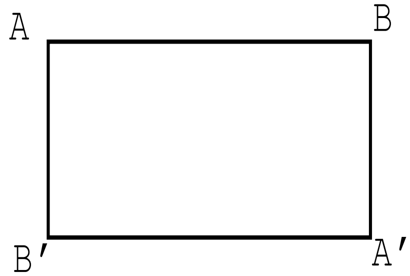


Figure 4: Beginning strip of paper

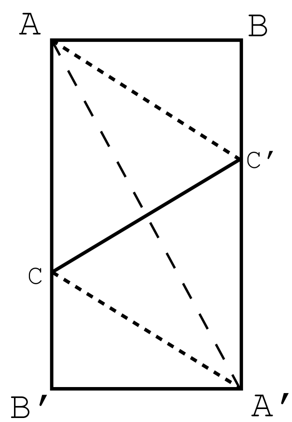


Figure 5: Construction for Mobius strip with ribbon length $1/\sqrt{3}$

Barr also cites [2] that Martin Gardner devised a method to make an indefinitely wide Mobius strip, or one with an arbitrarily small ribbon length. This can be done by simply creating an odd number of folds in the strip that run parallel to the core of the strip. This creates an accordion-style piece of paper. An end of this accordion figure can then be turned over and the ends may be joined to form the Mobius strip. This can be done for any odd number of folds, thus hypothetically creating a strip of indefinite width (if the paper is thin enough). An introduction of an even number of folds results in a figure that cannot be rejoined correctly.

Nevertheless, Barr's model offers a different view of ribbon length than that of Kauffman. In this model, folds do not need to occur from the core, so the width of the ribbon can be expanded without restraint. This decreases the ribbon length for knots as compared to Kauffman. However, ribbons with knotted cores cannot be created with this model as can Kauffman's construction. In terms of creating a knot (K, n), Kauffman's model is only be able to alter the core K, while Barr/Gardner's model can only alter the linking number n of the boundary with the core. Not every knot (K, n) can be constructed using Barr's model, and the introduction of many more mirrors is necessary to create a complicated (K, n) knot using Kauffman's model. The model below offers an alternate ribbon construction and subsequent notion of ribbon length.

4 Another Mathematical Model for Knotted Ribbons

Definition A *framed knot* is a smooth curve (the core) equipped with a continuous choice of orthogonal direction at each point on the curve.

This mathematical model creates framed knots in three-dimensional space. An orthogonal direction is chosen at each point on the smooth core, and the ribbon is formed by extending the width until the strip self-intersects. The core is chosen and the width is extended with enough twists in the ribbon to obtain the proper linking number. Using this continuous framed knot construction, any knot (K, n) can be formed by simply choosing the core K in three-dimensional space and expanding the ribbon with enough twists to create a linking number of n. Figure 6 shows a standard Mobius strip $(unknot, 1)$ that was constructed using this model without attempting to optimize ribbon length.

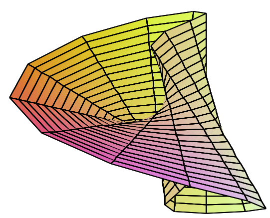


Figure 6: 3-Dimensional Mobius ribbon constructed with model

This definition of a framed knot is an adaptation of the concept of rope-length. Ribbon length is analogous to ongoing research on the minimum length to radial width of knotted tubes, called *thickness* [6]. No conjectures have been made yet for the thickness of any specific knots.

This mathematical model can construct knotted ribbons with an unknotted core that have minimal ribbon length. First, these knotted ribbons will be constructed with ribbon length of 1, and then it will be proven that 1 is in fact the smallest ribbon length that can be achieved for an unknotted core.

Example $rl(unknot, n) \leq 1$.

This optimal model for ribbons with unknotted cores is constructed as follows. The unknot core is formed in the xy-plane by parameterizing a rectangle of length $a+d$ and width $d+2l$ with semicircles of diameter d attached at each end. In the middle of the arch of each of the semicircles, a small straight edge is added of length $2l$, where l can be defined as some fraction of d, ($l = \frac{d}{10000}$, for example). In Figure 7 the core of a ribbon using this model is shown, and the length of this core is $2a + \pi d + 4l$. The length of the edge $2l$ is exaggerated in the figure for emphasis.

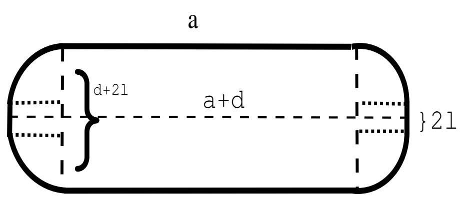


Figure 7: Ribbon core in xy-plane

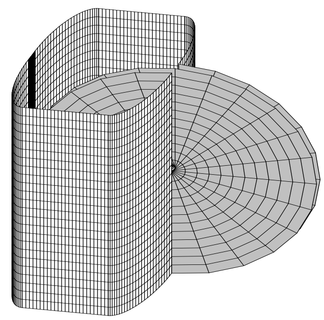


Figure 8: $3-D$ ribbon formed

It is on one of these straight edge portions of the core that the twisting occurs in the ribbon. On the straight edge, the width is expanded in both directions from the core on rotating vectors along the straight edge until it intersects the knot. This occurs at the opposite edge of length $2l$, and produces a ribbon of maximal width $2(a+d)$. Multiple twists can be introduced along this straight edge by simply altering the span of rotating vectors. On all other portions of the core, the width of the ribbon is expanded to length $2(a+d)$ from the core in the yz-plane, forming the ribbon. In Figure 8, the Hopf Link ($T_{2,2}$) is formed using this model, introducing one full twist to the ribbon. In general, this model creates a ribbon representing some $T_{2,n}, n \geq 1$ knot with core length $2a+\pi d+4l$ and width $2a+2d$. As d becomes arbitrarily small (i.e. the radius of the circle used to create the core is restricted) and hence l, a factor of d, becomes small, it can be seen that

$$\lim_{d \to 0} \frac{2a + \pi d + 4l}{2a + 2d} = 1.$$

Using this construction, an upper bound of 1 has been constructed for the ribbon length of all nontrivial framed unknots.

Remark It should be noted that with the exception of the Mobius strip, these framings with ribbon length 1 cannot actually be made with a square sheet of paper. As the number of half-twists increases, the length of the boundary of the framing also increases, and thus this only can be made with an infinitely thin ribbon.

A theorem will now be offered which states that the ribbon length, or minimal length to width ratio, of these knots actually equals one. First, the trivial case of an unknotted core framed with a trivial framing will be addressed. If we take an unknot core of length L to be just a simple closed curve on the $xy-$ plane, the width may be extended indefinitely in both directions from the core. Therefore, the ratio of length to width approaches 0, and so the ribbon length for this trivial construction is actually 0.

Theorem 1 *If U is the unknot and n is any nonzero integer, then $rl(U, n) = 1$.*

The continuous model created $(unknot, n)$ knots with ribbon length 1, thus creating an upper bound for the ribbon length of $(unknot, n)$ knots.To show that the ribbon length of theses knots actually equals 1, it must be proven that 1 is also the lower bound for ribbon length. This is synonymous to showing that the length of the core must be at least as large as the width of these ribbons. The proof of the theorem consists of several lemmas. First, the case for 2-dimensional planar cores will be shown.

Lemma 1 *The lower bound of ribbon length of any nontrivial planar framing of the unknot is 1.*

Proof Let K be a planar unknot, framed with any knot. Let $L = l(k)$ denote the length of the unknot core and let W denote the width of the ribbon. We want to show that $L \geq W$. In this framed ribbon, W is extended in both directions perpendicular to K. The frame must necessarily puncture the disk that K bounds at least once to form the knot itself. For all points P on K, let n be the line that passes through P normal to K. Let $C(K, P)$ be the set of points on $n \cap K$ (other than P), and let Q be the closest of these points to P. Define $R = \{|\overline{PQ}| : C(K, P) = Q\}$. Then the maximum that the width can be extended in one direction from K is $\max R$, so $\frac{1}{2}W \leq \max R$. Also, since $\overline{PQ}$ is the smallest distance from P to Q and K runs from P to Q and back to form the unknot, it follows that

$$L = l(K) \geq 2|\overline{PQ}| = 2\max R \geq 2(\frac{1}{2}W) = W. \square$$

Next, using the topological concept of a *convex hull*, it will be shown that the ribbon length for any nontrivial framed unknot is at least 1, and therefore equal to 1. The proof will actually generalize this to any nontrivial framing of any core, not just unknotted cores.

Lemma 2 *The ribbon length of any nontrivial framing of any core is at least 1.*

Proof Let K be the core of a nontrivially framed knot in three dimensional space. It will be shown that the arc length $\mathcal{L}(K)$ of K is at least as large as the width of the frame of the ribbon, W. To do this, the convex hull of the core K must first be defined, as given by Valentine [7]. The *convex hull of K*, or *conv* K, is the intersection of all convex sets which contain K. A set $S \subset$ linear space L is said to be *convex* if for each pair of points $x, y \in S$, the line segment xy joining x and y is also in S. This is analogous to the familiar concept of convexity. Let d denote the diameter of a set K, which is $\sup_{x\in K, y\in K} ||x-y||$ or the maximum width of K, as defined by Lay [5]. $K \subset$ *conv* K, so all points on K are at most d apart. The ribbon width is expanded as far as possible from K until self-intersection occurs. The core K bounds a surface in 3-dimensional space that the frame (the width) must necessarily puncture at least once to form the nontrivial framing. In the case of the unknotted core, this surface is simply a disk. For knotted cores, this surface is called a *Seifert surface* [3]. This surface will be punctured either on its boundary or on its interior, leading to two cases.

Case 1: The surface is punctured on its boundary

Now, the boundary of this surface is simply the knot itself, which is known to be inside the *conv* K. Therefore, the ribbon width can be extended in that direction at most distance d from the knot, or $\frac{1}{2}W \leq d$. Since d is the distance between two points which are both on K, K must travel to and from these points to be created. Because the straight-line d is the smallest distance between any two points, it follows that

$$\mathcal{L}(K) > 2d \geq 2(\frac{1}{2}W) = W.$$

Case 2: The ribbon width punctures the interior of the surface that K bounds

Now, $\mathcal{L}(K)$ must travel more than $2d$ to be formed, so let $\epsilon = \mathcal{L}(K) - 2d$. The maximum distance from any point on the surface to the convex hull may be restricted to no more than $\frac{\epsilon}{2}$; or, in other words, the surface may be restricted to lie within the union of $\frac{\epsilon}{2}$ balls on the boundary of the *conv* K. To do this, the region between the *conv* K and the farthest point away on the surface must be rescaled with an isotopy *outside* of the *conv* K, an action which leaves the *conv* K unaltered. This ensures that the puncture occurs within a distance of $\frac{\epsilon}{2} + d$ from the knot itself. This puncture creates the width in one direction from the core, so the width of the ribbon itself is twice this distance. Therefore,

$$W \leq 2(\frac{\epsilon}{2} + d) = \epsilon + 2d = \mathcal{L}(K).\square$$

It has been shown that the ribbon length for all nontrivial framings of the unknot is 1, and a lower bound of 1 has been shown for the ribbon length of nontrivially framed knotted cores. If a knotted core is framed trivially, the frame does not have to puncture the Seifert surface to be formed. However, a lower bound can still be formulated for these framed knots. It will now be shown that a lower bound on the ribbon length for a trivially framed knotted core is 1.

Theorem 2 *The ribbon length of a trivially framed knotted core is at least 1.*

Proof To prove this, it must first be shown that in a trivial framing, the width of the knot will intersect another part of the knot. This argument is similar to one found in a paper by Adams, et al [1]. Assume to the contrary that the trivial framing will not intersect K. Then at each point on K, the frame can be extended indefinitely. But an investigation of the union of the rays that compose this indefinite width along with a point at infinity yields an immersed open unpunctured disk with K as its boundary. This would force K to be an unknot, but it is assumed that K is a knot. Therefore, the trivial framing must intersect K. The point where this occurs is the maximum width that the ribbon can be extended in one direction, and is no farther away than d. As in the former proofs, this implies that $\frac{1}{2}W \leq d$. Since d is the distance between two points which are both on K, K must travel to and from these points to be created. Because the straight-line d is the smallest distance between any two points, it follows that

$$\mathcal{L}(K) \geq 2d \geq 2(\frac{1}{2}W) = W. \square$$

5 Discussion

This continuous model described provides a very different view of ribbons and ribbon length than both Kauffman and Barr. With Kauffman's models, the linking number n of a knot (K, n) can only be altered by the introduction of more mirrors, subsequently increasing the ribbon length for many (K, n) knots. With the continuous model, the width of the ribbon is expanded maximally from the core without the introduction of folds. Therefore, any amount of twists can be introduced in the construction to alter n without increasing the ribbon length of the knot. This model improved the ribbon length of Kauffman's model. However, ribbons created using the continuous model anomalistically are not paper constructable, while Kauffman's are.

With this continuous model, all twists that occur in the ribbon originate from the core itself, as in the definition of a framed knot. Therefore, each twist directly alters the linking number n of the boundary with the core. The folds of both Barr's model and the one offered by Gardner do not intersect the boundary of the ribbon frame itself and are not expanded in *one direction* from the unknot core, because other folds are introduced. These folds do not alter the linking number n and violate the notion of a framed knot. Since these models violate the definition of a framed knot, the notions of ribbon length described here are essentially different.

A ribbon length of 1 has now been established for all nontrivial framed knots with the unknot as core using the continuous model. In addition, a lower bound of 1 has been established for all ribbons with knotted cores, regardless of framing. It is conjectured that a better lower bound can be found for the ribbon length of framed knotted cores, as intuitively more ribbon must be necessary to form more complicated knotted cores. Further research must be completed on this topic to find a better lower bound.

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