Dowker Notation and Arc Presentation

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Abstract

In this paper we will discuss Cubic Lattice Questions. We will learn a detailed definition and description of what Dowker Notation is and how to use it. We will discuss how to get the Dowker Notation from a diagram of a knot, how to go from the Dowker Notation and create a diagram of a knot, and how to transform a knot diagram to a representation of the same knot on the cubic lattice. Dowker Notation is a very useful tool in drawing diagrams of knots and then placing them on the cubic lattice. We will also see how to go directly from a diagram of a knot to a cubic lattice representation of that knot. Lastly, we will look at the processes of arc presentation of a knot. Beginning with the arc presentation of a knot, we will derive a formula for the upper bound for ropelength on the cubic lattice.

Keywords: Cubic Lattice, Arc Presentation, Knot Diagrams, Dowker Notation, Ropelength

1 Introduction

This paper will discuss three different areas of knots related to the cubic lattice.

The first is Dowker Notation, also known as Dowker-Thistlethwaite notation is a notation named after Clifford Hugh Dowker and Morwen Thistlethwaite who developed it. They built their notation after refining a notation created by Peter Guthrie Tait. Dowker notation is a very useful tool. You can begin with a drawing of a knot and find the Dowker Notation for that knot. From there you can then draw the Dowker Diagram to represent that knot. From the Dowker Diagram you can map the knot to the cubic lattice. This will be shown and explained step by step in this paper.

Secondly, we will show how to go directly from a diagram of a knot to the cubic lattice. While this isn't the most efficient way to represent a knot on the cubic lattice, it is the most direct way. We had hoped that one of these two ways of drawing knots of the cubic lattice would prove to be systematic enough to come up with a general formula for the upper bound of ropelength on the cubic lattice. Unfortunately, neither of these provided sufficient structure to algorithmically place a knot on the cubit lattice.

The third and final way we will investigate is Arc Presentation. This is a much more systematic representation of a knot and we were able to derive a general formula for the upper bound of ropelength of a knot. Putting a knot on the cubic lattice is an important representation of a 3-dimensional drawing of a knot. The grid arc presentation of a knot is almost identical to the cubic lattice drawing, but it does not show the 3-dimensional aspect around each crossing. The formula we have accounts for this and will be explained and defined in this report.

Here are some important definitions to begin with:

Definition 1.1 (Arc Presentation) *An arc presentation is a representation of a knot in grid form. It shows the knot on an x,y plane type graph and every edge is straight. An arc presentation of a link, L is an embedding of L in finitely many pages of the open-book decomposition so that each of these pages meets L in a single simple arc.*

Definition 1.2 (Cubic Lattice) *A representation of a knot in 3-space in a grid pattern.*

Definition 1.3 (Cubic Lattice length) *The length of rope required to place a knot on the cubic lattice.*

Definition 1.4 (Injectivity Radius) *The radius of the largest tube with the curve at its core.*

Definition 1.5 (Rope length) *The length of the curve divided by its injectivity radius.*

Important Note for Reference: One half of the ropelength is less than or equal to the cubic lattice ropelength.

Definition 1.6 (Dowker Notation) - *Dowker Notation is a sequence of even integers that represent crossings of a knot. These even numbers can be paired with odd numbers sequentially to see what numbers share a crossing on a knot. From this information you can draw a diagram representation of any knot and then you can draw a knot on the cubic lattice.*

2 Examples and Algorithms for Dowker Notation

To begin, we start with a drawing of a knot and get the Dowker Notation:

Algorithm 2.1 (To Get Dowker Notation From a Drawing of a Knot)
1. Chose a direction you want to travel on your knot and chose any undercrossing and label it crossing 1.

2. Continue going in that direction and label the crossings sequentially. (Note- you will have an even and an odd number at each crossing)

3. If there are any even numbers that represent an undercrossing, label them with a negative number.

4. List the odd numbers in order on top of a chart and below each odd number, write the even number that it shares a crossing with.

5. The even numbers in this order (in this example 12 8 -10 -14 16 -6 2 -4) represent Dowker Notation.

(Both the knot and the chart are shown below in an example)

From this knot, we get the following Dowker Notation:

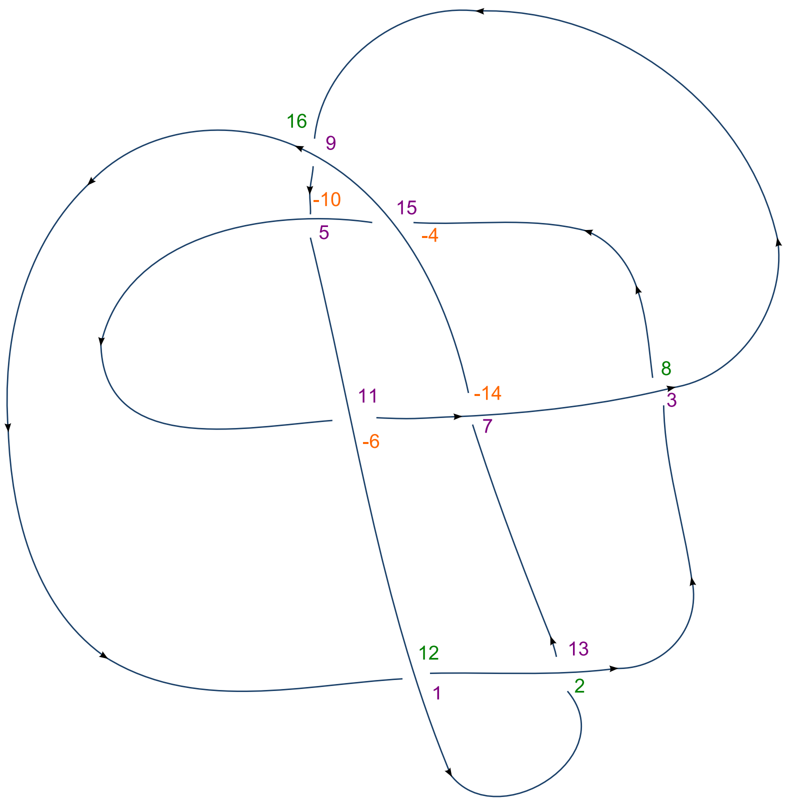

Figure 2.2

1	3	5	7	9	11	13	15
12	8	-10	-14	16	-6	2	-4

Next, we use Dowker Notation to draw the Dowker Diagram:

Algorithm 2.3 (To Draw a Dowker Diagram from Dowker Notation)
1. Begin with the table shown in the example above- with the odd number sequentially listed above the even number each odd is paired with in the crossings of the knot.

2. Draw a horizontal line to represent the beginning of your knot and, along that line, draw crossings in order numerically until you reach a number that has already been represented in a previous crossing. We will call this initial horizontal line, the main line.

3. Once you reach such a crossing, loop back and connect the two numbers.

4. Continue looping and connecting the numbers sequentially until all numbers have been connected in order numerically.

Note- Some numbers may not be on the original line. It is okay to have a crossing off of that line if necessary.

5. Connect your final number with number 1 to complete the knot.

(Note- This is shown below in an example drawing of the knot referred to on the previous page.)

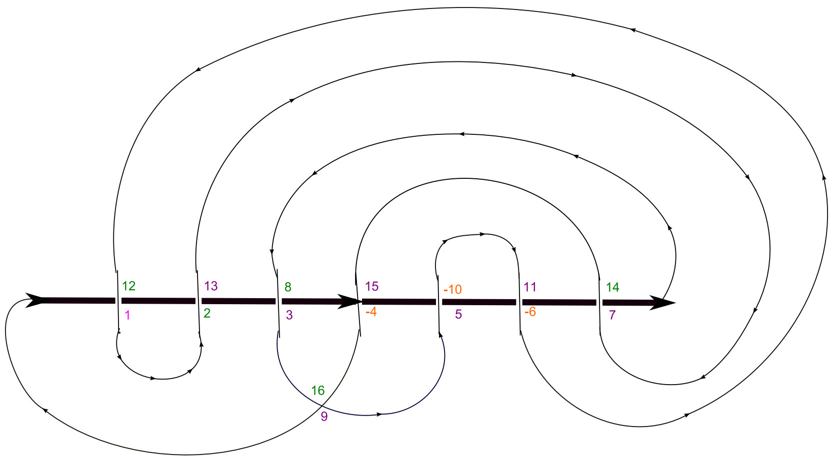

Figure 2.4

There are a couple of observations we can make here:

Remark 2.5 *Since we have a systematic drawing of our knot, you won't ever cross your first "loop" from your main line to the first number out of the sequence (in this example, this loop is from 7 to 8.) This line will never be crossed because the numbers that exist prior to the crossing in question (in this case 3 and 8) have all already been accounted for in the main line. Since the crossings from 4 to 7 have already been drawn, we won't ever have to cross that first loop to get to another crossing.*

Remark 2.6 *Secondly, you don't not have a crossing off of the main line of numbers in every diagram, this only happens in certain cases like the one we had above. If two of the first half of numbers are paired with each other in a crossing then the second half of the numbers will also have a pairing with each other. Similarly, if there are two pairs of numbers of crossings in the first half, then there will be two pairs shared in the second half as well, and so on. In our example, this happens with the pairings of 3 and 8 and then 9 and 16.*

Remark 2.7 *Also, the negative and positive numbers are only required in a non-alternating knot. When we have an alternating knot, the numbers can all be positive because the negative numbers represent even numbers that are attached to undercrossings. In a perfectly alternating knot, the 1 crossing will be an undercrossing and every even number will be an overcrossing and will therefore be positive.*

Then, we can square off the Dowker Diagram to place the drawing on the Cubic Lattice:

Algorithm 2.8 (To Draw a Diagram on the Cubic Lattice from a Dowker Diagram)
1. Using your Dowker Diagram as the model, square it off as best as possible. Redraw it making all of the edges square and all of the lines straight. (It might help to do this on graph paper.)

2. The cubic lattice uses sticks of length 1 unit to represent everything, so make dots to represent the ends of each stick at equal intervals. Forexample, there should be one stick in between each crossing and so forth.

3. You may have to adjust crossings that are not on the main line to line them up with the sticks, but they should fit nicely above or below the main line.

4. Make sure there is at least one stick length in between each space and that nothing uses the same line more than once.

5. To transfer the diagram into a 3-D effect, add sticks going upward on either side of each upper crossing.

Shown below is the basic cubic lattice drawning and then the cubic lattice drawing with the 3-D effect to show the extra sticks required to go up for overcrossings.

Regular Cubic Lattice Drawing:

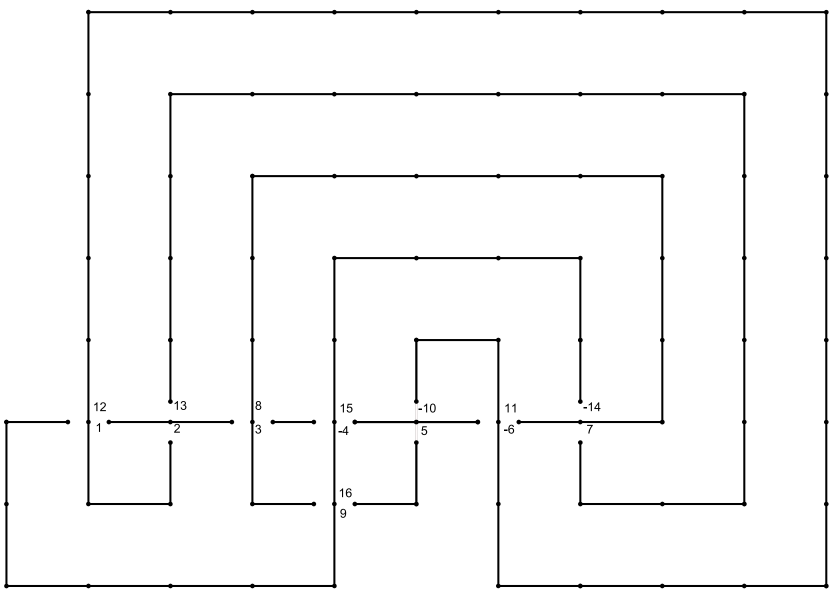

Figure 2.9

3-D Cubic Lattice Drawing:

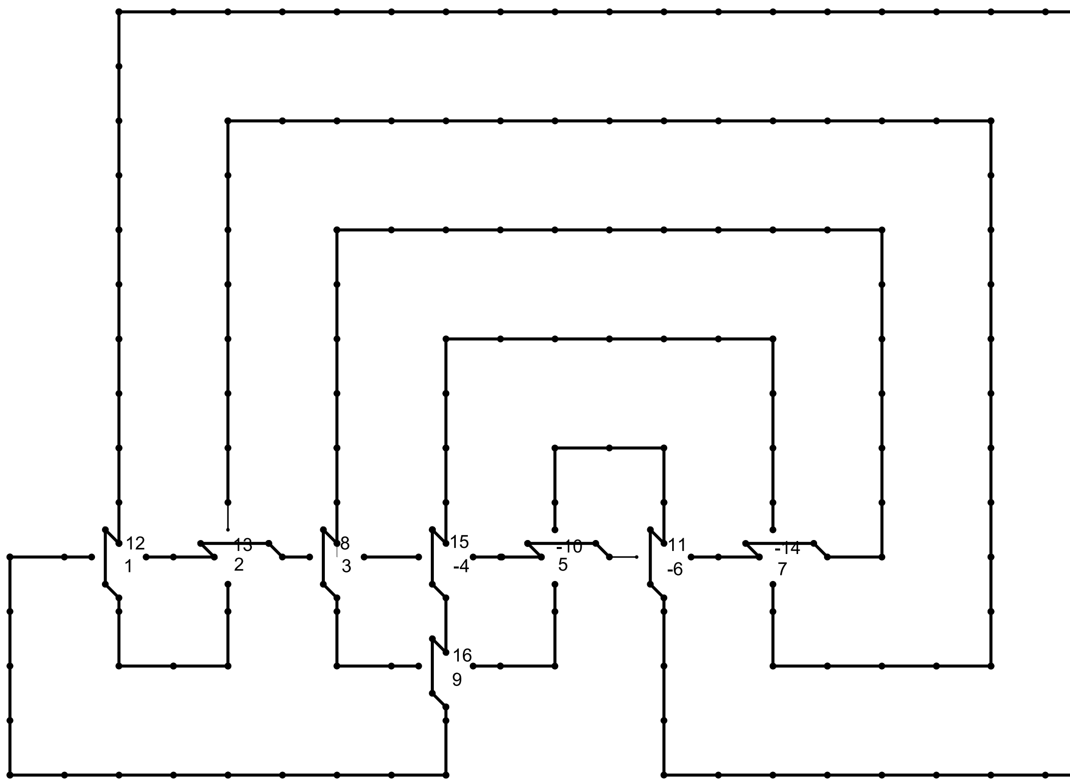

Figure 2.10

Now we will follow the same steps for a couple other knots:

The basic alternating trefoil is a three crossing knot that alternates perfectly. Since it alternates, negative numbers will not be required in Dowker Notation. This implies that every even number is an overcrossing.

Here is the trefoil in the original knot drawing and labeled as mentioned in Algorithm 1.1.

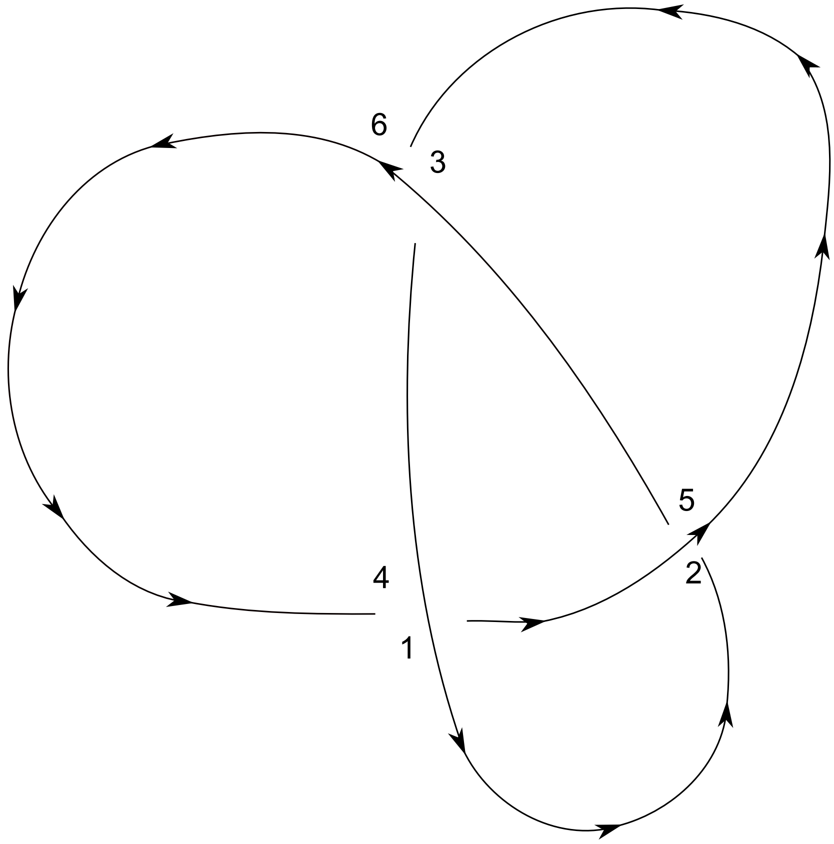

Figure 2.11

Using this knot, we obtain the following Dowker notation:

1	3	5
4	6	2

Next we can use our Dowker Notation to draw our Dowker Diagram using Algorithm 1.2:

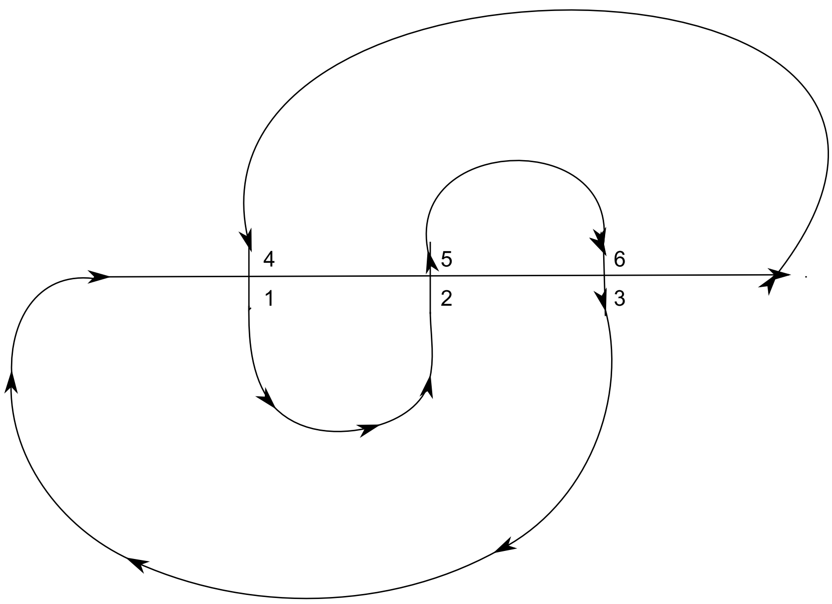

Figure 2.12

We can square the picture off and place it on the cubic lattice using Algorithm 1.6:

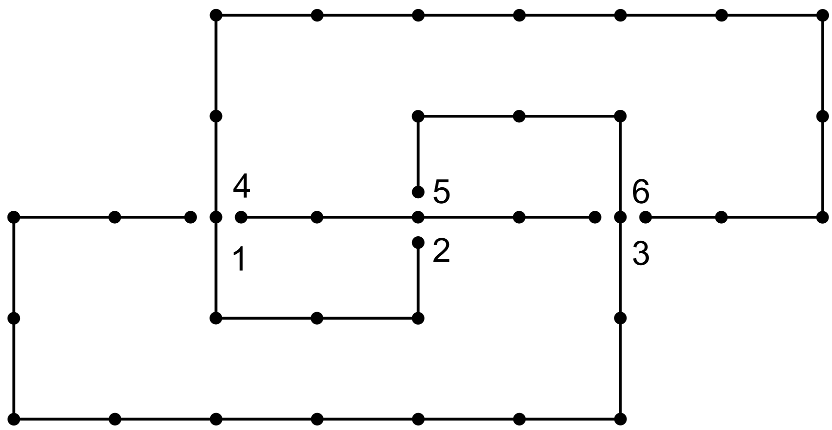

Figure 2.13

Lastly, we can covert this to the 3-D Cubic Lattice in order to get the upper bound for the ropelength.

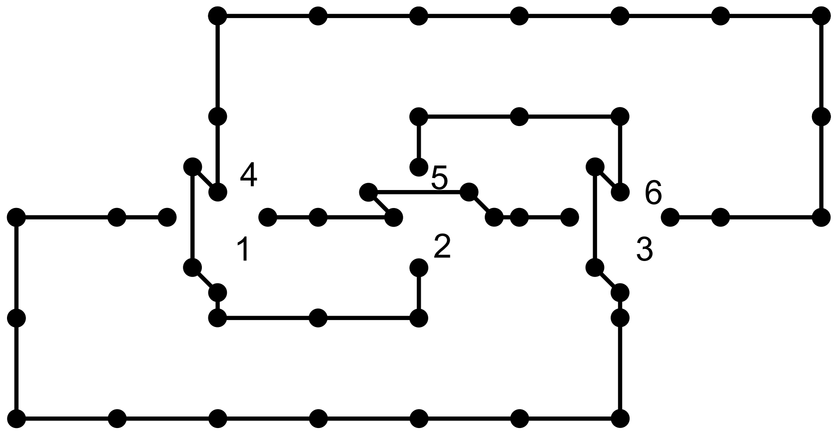

Figure 2.14

Next we will look at another example beginning with the following Dowker Notation:

1	3	5	7	9	11	13	15	17	19	21	23	25
4	8	12	2	16	6	22	18	10	26	24	14	20

Using this notation, we will get the following Dowker Diagram (Algorim 1.2):

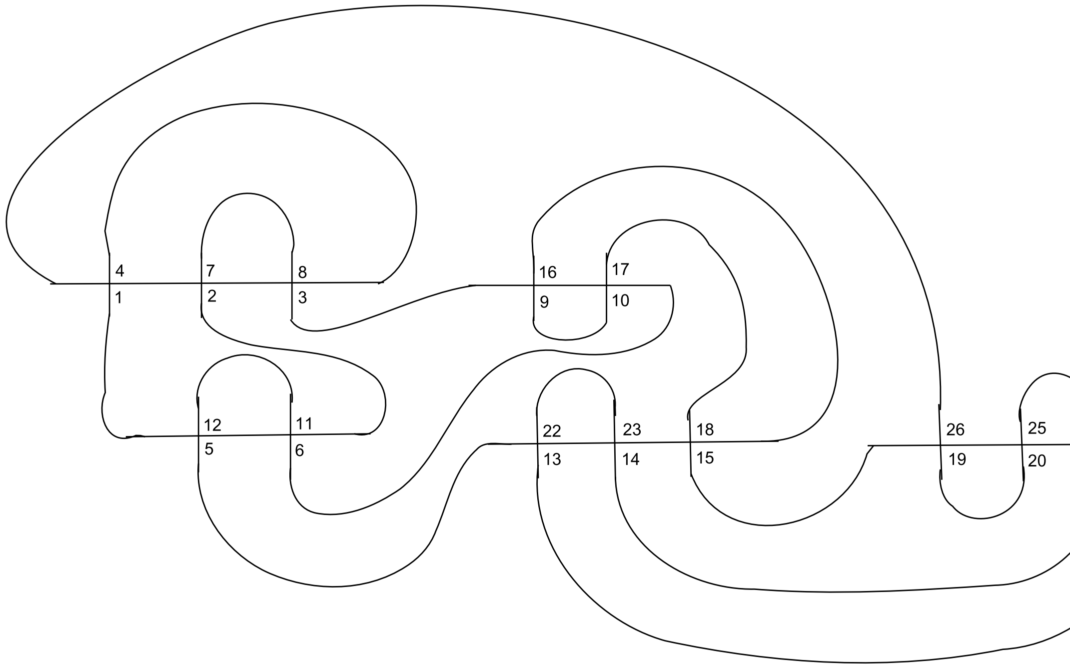

Figure 2.15

From that diagram, it can be squared off and placed on the cubic lattice (Algorithm 1.6):

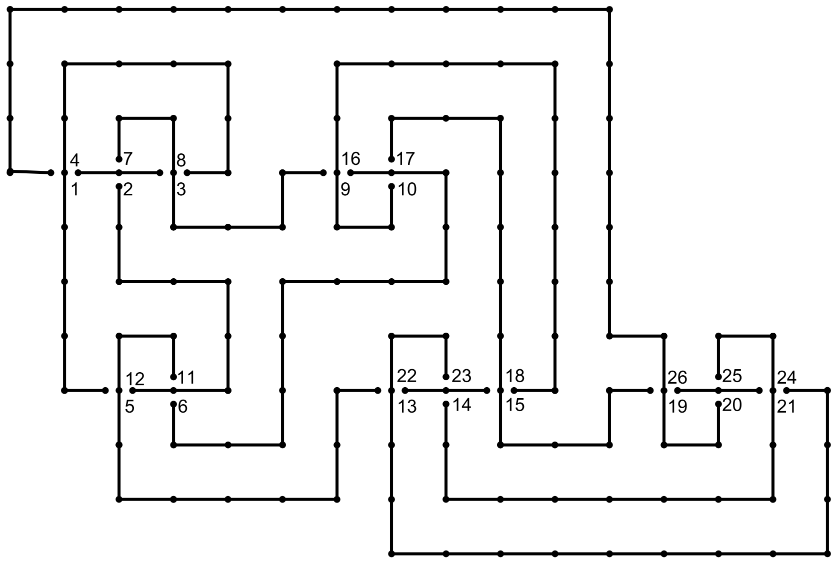

Figure 2.16

Finally, we can place it on the cubic lattice in 3-D:

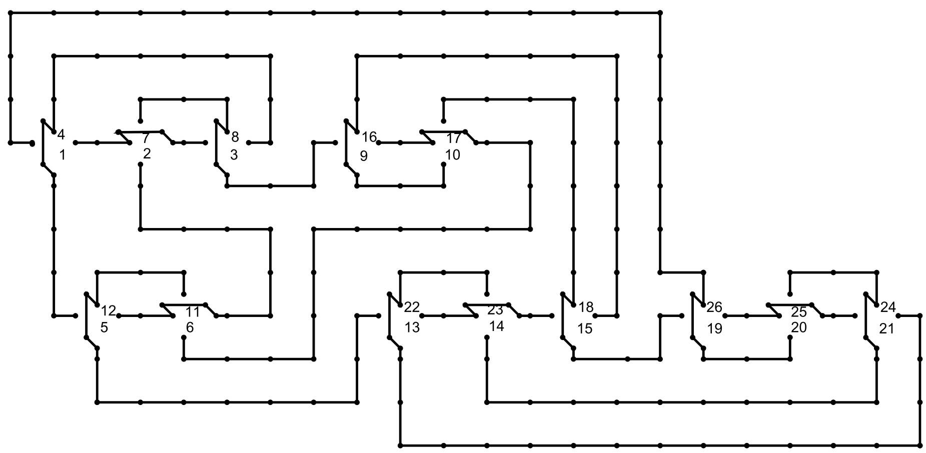

Figure 2.17

Unfortunately, Dowker Notation was not succesful in finding a systematic way to put knots on the cubic lattice. Now we will try to go directly from a knot diagram to the cubic lattice. Below is the representation of a knot and then the cubic lattice drawing obtained from that knot:

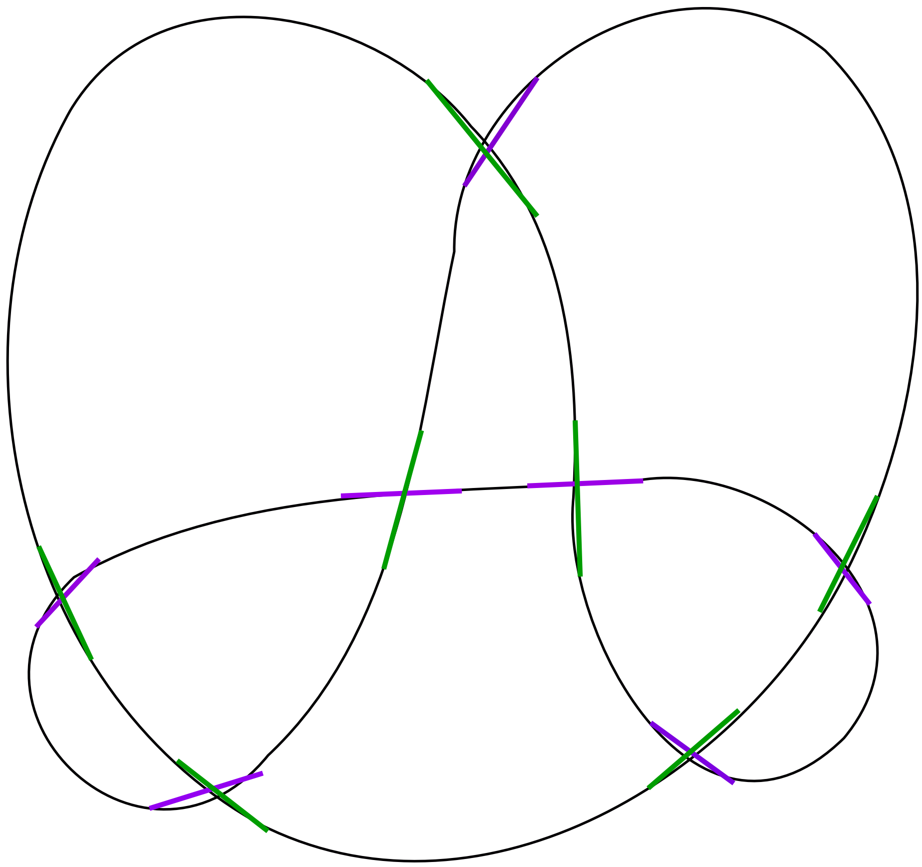

Figure 2.18

The green and purple crossings in the picture above will help us as guides to transferring the drawing over to the cubic lattice. See the corresponding crossings below:

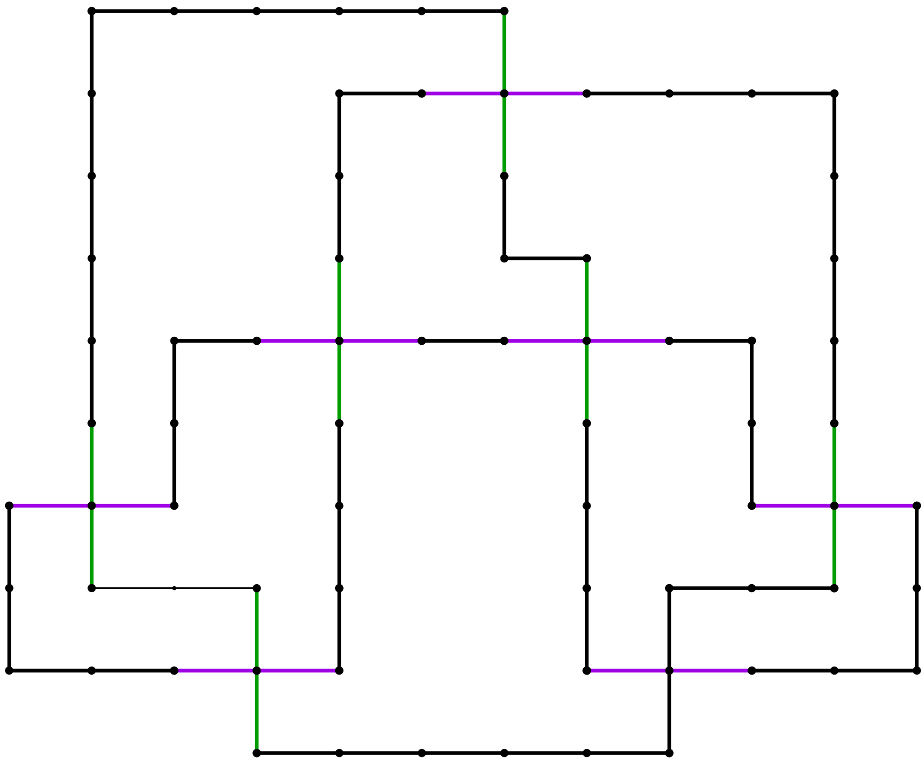

Figure 2.19

On the next page you will see the algorithm to go from the drawing of a knot to the cubic lattice diagram as shown in this example.

Algorithm 2.20 (To Create a Drawing on the Cubic Lattice from a Knot Diagram) *1. Begin by marking each vertex of your knot with two lines. It helps if you are able to use two different colors to differentiate between the two. These two lines (shown in purple and green in our example) will represent the perpendicular crossing on the cubic lattice of each vertex.*

2. Pick one of these vertices and draw it on the cubic lattice. This is where you will begin transfering the drawing to the cubic lattice.

3. Approximate where the other vertices should go in relation to this original vertex.

Note- You may have to move things around to be sure there is one unit length in between each line, but this is a good starting off point.

4. Connect all of the vertices. Be sure to only use straight lines along the cubic lattice, either going horizontally or vertically. Do not draw any diagonal lines.

5. Once you are finished you can alter the spacing so that everything is one until apart. You want to use the minuimum amount of space needed without overlapping any lines.

This method, like Dowker Notation, was also not sytematic enough to allow for counting the number of edges needed for the upper bound for ropelength for a cubic lattice knot.

3 Arc Presentation

We then turned to a third method of representing a knot.

Arc Presentation is one way to describe a knot. The previous two ways that were attempted were unseccesful in putting a knot on a cubic lattice systematically, but using the Arc Presentation will be much more systematic. This section will cover describing how this will work.

Definition 3.1 (Arc Presentation) - *An Arc Presentation is a representation of a knot in grid form. It shows the knot on an x,y plane type graph and every edge is straight. An Arc Presentation of a link, L is an embedding of L in finitely many pages of the open-book decomposition so that each of these pages meets L in a single simple arc. This presentation can be used to get a pair of permutations to represent the knot and then can also be used to graph a knot on the cubic lattice and get an upper bound for the ropelength of that knot.*

4 Introduction

A 4 crossing knot:

Shown below is a four crossing alternating knot. This is the typical presentation of the drawing of a knot.

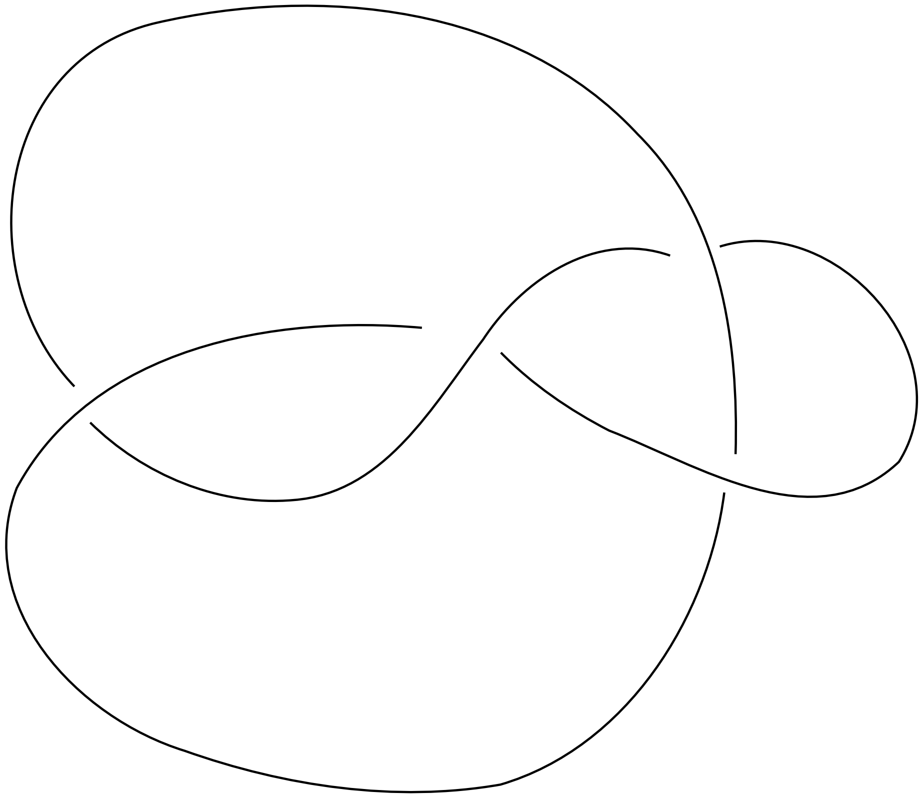

Figure 4.1

An arc presentation of that 4 crossing knot:

Shown below is the Arc Presentation of the four crossing knot shown previously. Once we are given this format, we can move forward to get the pair of permutations. Also, this grided arc presentation looks very much like the cubic lattice, therefore the transition to the cubic lattice and ultimately to the ropelength will be fiarly simplistic and systematic.

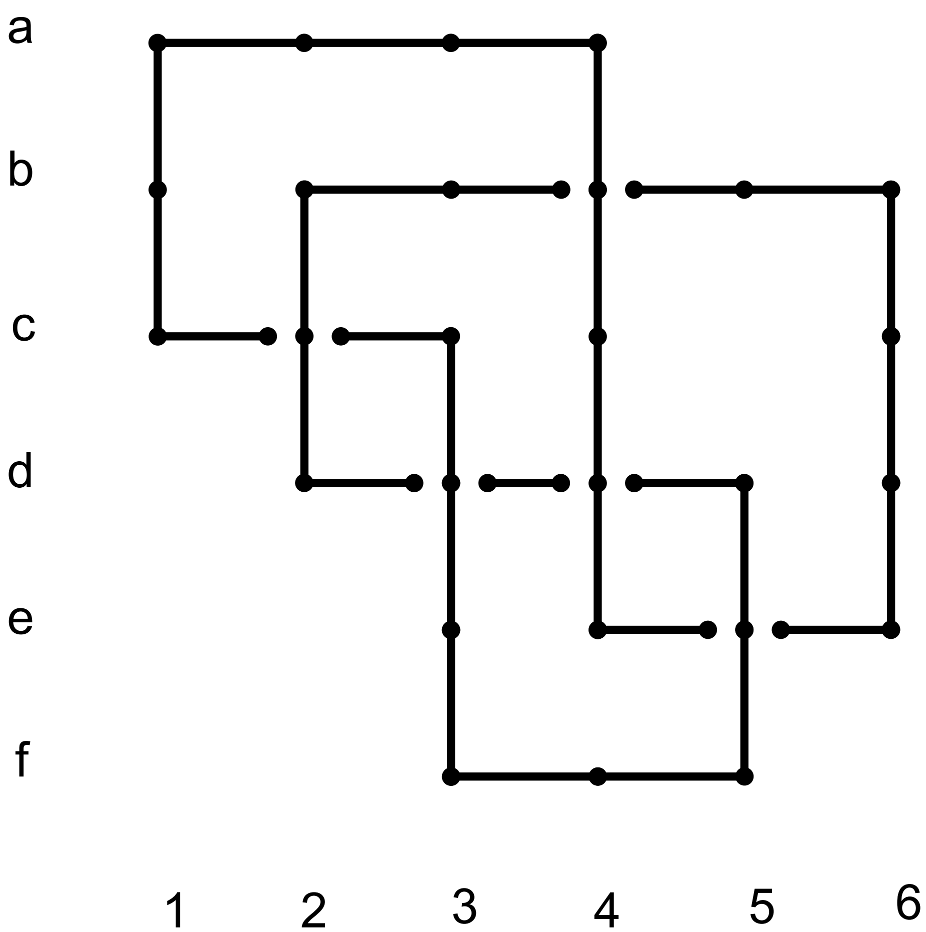

Figure 4.2

The arc presentation in axial form of the 4 crossing knot

This is the closest presentation that can be included in this report to show a knot embedded in half planes. The line down the middle of the drawing represents the center of revolution for the planes, which we will call the axis of the arc presentation for the knot. Think of each arc as lying on a half plane whose boundary is the axis. You can see the 3-D effect by looking at the arcs on either side and seeing which arc cross over the other arcs. For example, this drawing has 6 arcs, each lying in it's own half plane. Arcs 1, 2, and 3 are all on the right and arcs 4, 5, and 6 are on the left. So, arc 1 is in front of arc 2 which is in front of arc 3. Arc 6 is in front of arc 5 which is in front of arc 4. You can imagine the half planes containing these arcs rotating coutner clockwise around the axis and creating a full three-dimensional circle.

To go from the grid format in figure ********number all figures Jen********* the the axial form below is a few simple steps. First, we label the vertical side labeling each horizontal line alphabetically. In this case we begin at the top with "a" and go down labeling each horizontal line until we get to "f." We will label the vertical lines similarly, but this time we will label them numerically begining at the far left and moving right. In this example we will go from "1" to "6." Each vertical line will become an arc in the axial presentation. Each horizontal lines represent where the arcs cross the axis of rotation. Once we have the labeling completed and we have our center axis drawn, we label points on the line from a to f. We will now connect these lettered points with the corresponding arcs. Begin by looking at the grid drawing to see which two letters are connected at 1. In this case, 1 goes from "a" to "c" and 2 goes from "b" to "d" and so on... We make sure that we have half of the planes on one side of the center line and the other half on the other side. We also make sure that the crossings that should be in front or behind of the others are properly represented. In this specific example below, 1 is in front going from a to c, 2 is behind 1 going from b to d, 3 is behind 1 and 2 going from c to f and those three are on the right. On the left, 6 is in front going from b to e, 5 is behind 6 going from d to f, and 4 is behind 5 and 6 going from a to e. Using this format, we can draw the axial form of any knot.

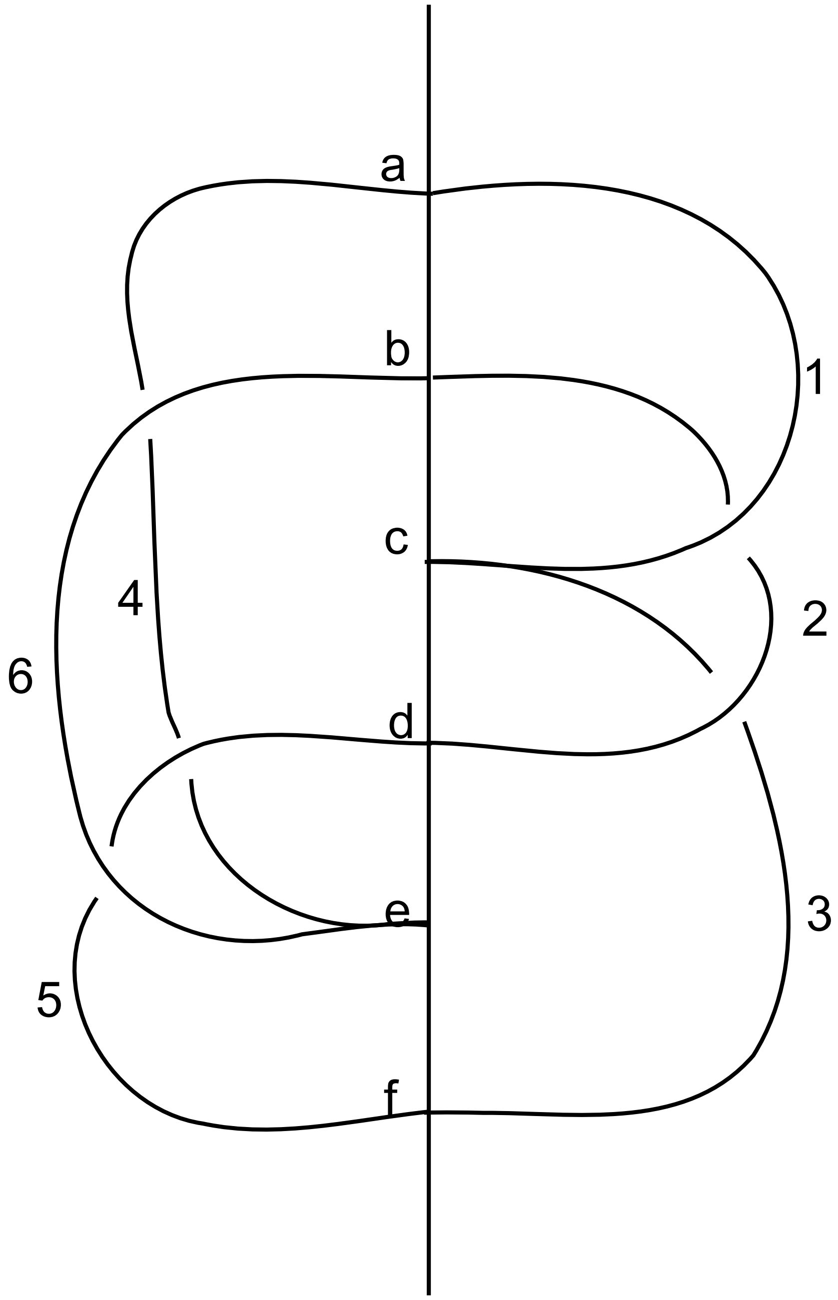

Figure 4.3

A third way to incorporate Arc Presentation of a knot is to use a pair of permutations created from referencing the axial drawing *****enter figure number Jen******. Beginning at a, we will follow the arcs in order until we get back to a. The top permutation will involve the leters and will represent the vertical length. (This will be used later to obtain the ropelength required for the vertical sides on the cubic lattice.) In this example we see that a goes to c which goes to f which goes to d which goes to b which goes to e which goes back to a. This is how we get (acfdbe.) We similarly look at the numbers for the horizonal length and use this to make our second part of the pair of permutations. Coming into a we see we are using number 4 which then crosses through a into number 1 which crosses c and goes into number 3 which crosses f and goes into number 5 which crosses d and goes into number 2 which crosses b and goes into number 6. Number 6 crosses e which leads us back to number 4 where we started, so we know our second part of the pair is (413526.) This is properly shown below:

The pair of permutations of the knot

$$(acfdbe)$$
$$(413526)$$

A 6 crossing knot:
Here is a similar example to the one described in detail on the previous pages using a six crossing knot. Here is the general depiction of the six crossing, alternating knot:

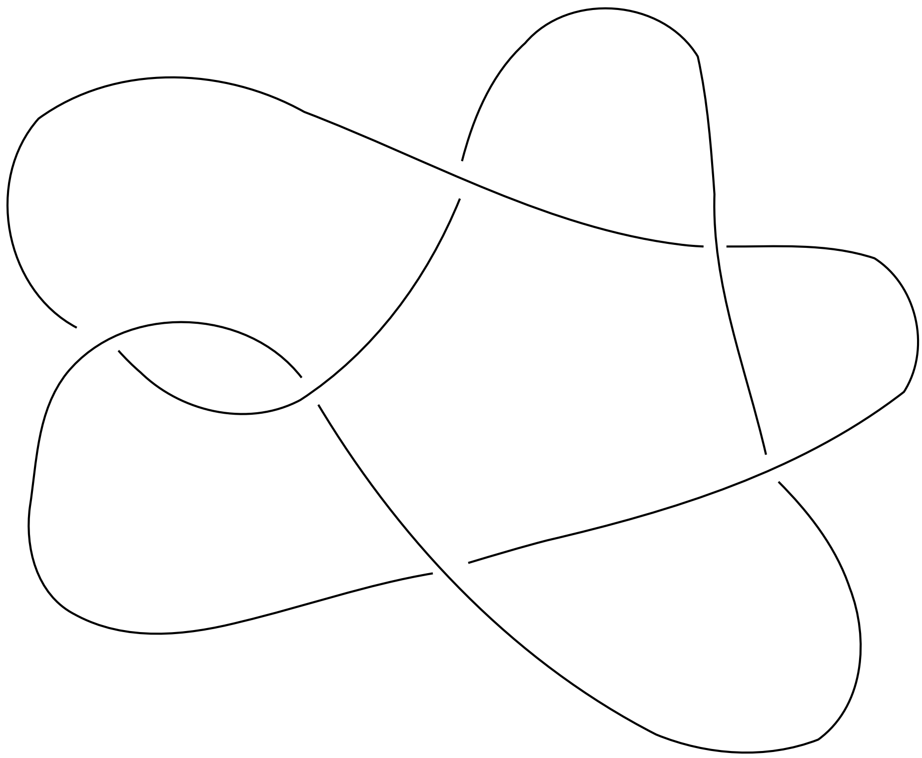

Figure 4.4

Here is the grid representation of the Arc Presentation of that knot:

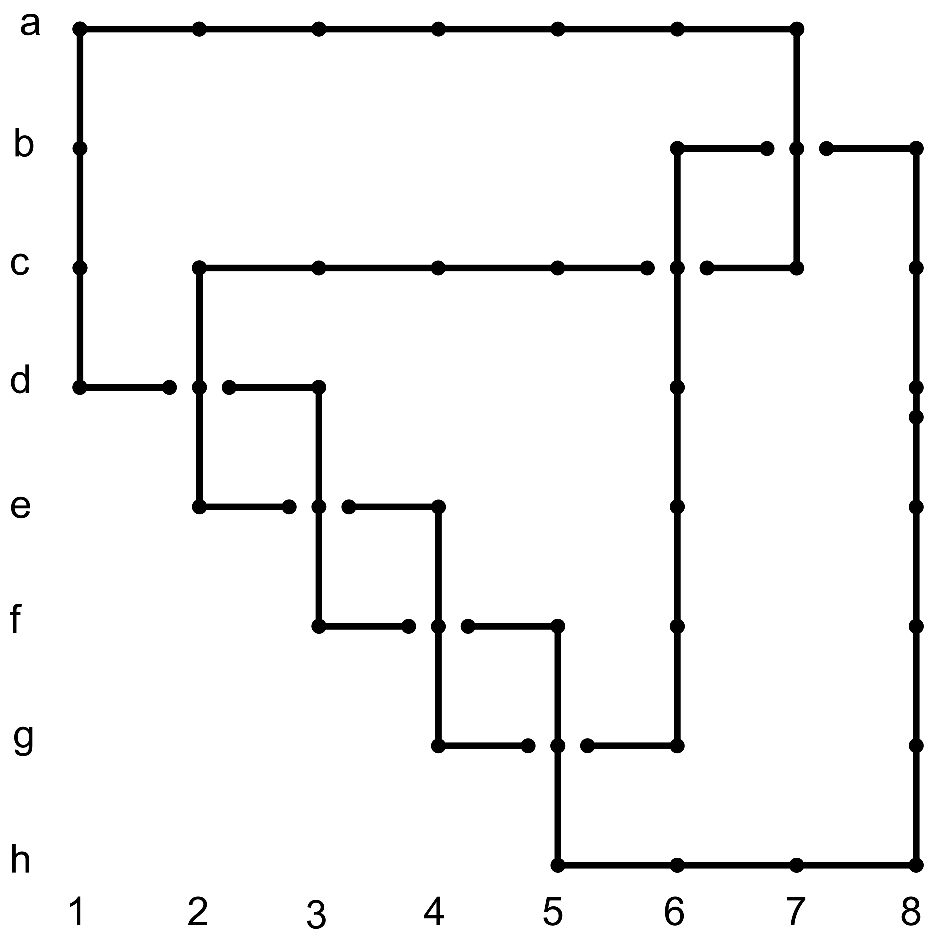

Figure 4.5

The Arc Presentation in axial form of the same 6 crossing knot obtained from the grid representation:

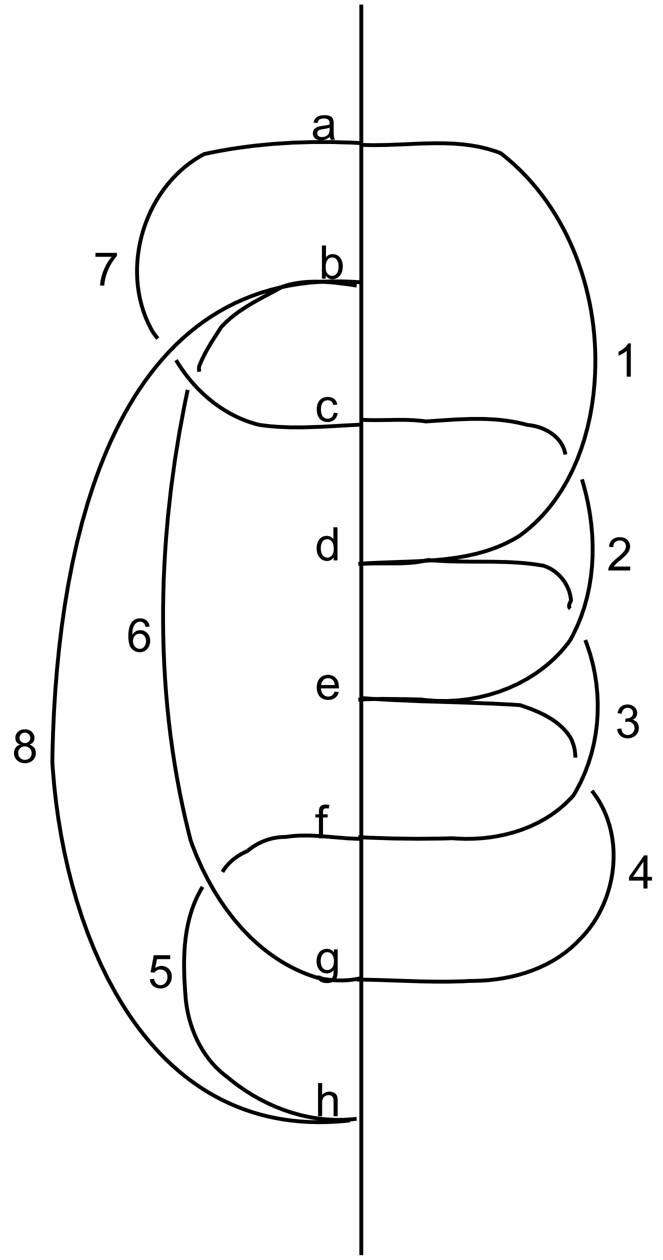

Figure 4.6

The pair of permutations of the knot obtained from the axial form of the Arc Presentation:

$$(adfhbgec)$$
$$(71358642)$$

Using the pair of permutation described and shown in section 5, we can derive the general formula for the upper bound of rope length. By converting the letters in the pair into the corresponding numbers (**a** is **1**, **b** is **2**...), we can find the rope length for the vertical pieces of the cubic lattice. Given the general vertical permutation to be represented by $(v_1v_2v_3...v_a)$, the formula will be:

$$|v_2 - v_1| + |v_3 - v_2| + ... + |v_1 - v_a|$$

Similarly, given the general horizontal permutation to be represented by $(h_1h_2h_3...h_b)$, the formula will be:

$$|h_2 - h_1| + |h_3 - h_2| + ... + |h_1 - h_b|$$

To account for the three dimensional aspect of each crossing, we must add 2(c) where c is the number of crossings to our formula. This makes our final formula:

$$(|v_2 - v_1| + ... + |v_1 - v_a|) + (|h_2 - h_1| + ... + |h_1 - h_b|) + 2(c)$$

In the 4 crossing example shown, our formula would be:

$$(|3-1| + |6-3| + |4-6| + |2-4| + |5-2| + |1-5|)+$$
$$(|1-4| + |3-1| + |5-3| + |2-5| + |6-2| + |4-6|) + 2(4) = 40$$

In the 6 crossing example shown, our formula would be:

$$(|4-1|+|6-4|+|8-6|+|2-8|+|7-2|+|5-7|+|3-5|+|1-3|)+(|1-7|+|3-1|+|5-3|+|8-5|+|6-8$$