ABSTRACT. This research discovers a new method to finding invariant subspaces. Invariant subspaces are important for finding the shape of matrices within the structure group. This research also discovers an example of a finite structure group in the 3-dimensional case. This method uses what is known about sectional curvature and links it to the idea of invariant subspaces.

1. INTRODUCTION

Definition 1.1. An algebraic curvature tensor $R$ is a function on a vector space $V$ defined as $R : V \times V \times V \times V \to \mathbb{R}$ satisfying, for all $x, y, z, w, v \in V$ and $c \in \mathbb{R}$, the following properties:

1. Multilinearity:
   \[ R(cx + y, z, w, v) = cR(x, z, w, v) + R(y, z, w, v), \]
2. Antisymmetric in the first two slot:
   \[ R(x, y, z, w) = -R(y, x, z, w), \]
3. Symmetric in the (1,2)-(3,4) slots:
   \[ R(x, y, z, w) = R(z, w, x, y), \]
4. Bianchi Identity:
   \[ R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \]

In the case that we evaluate $R(e_i, e_j, e_k, e_l)$, then we write $R_{ijkl}$, where \{e$_1$, e$_2$, ..., e$_n$\} is a basis. The use of this function $R$ is important in the study of Differential Geometry.

Definition 1.2. In a vector space $V$, a symmetric bilinear form $\varphi : V \times V \to \mathbb{R}$ where, for all $x, y, x_1, x_2 \in V$ and $c \in \mathbb{R}$, $\varphi$ is:

1. Symmetric: $\varphi(x, y) = \varphi(y, x)$,
2. Linear in the first slot: $\varphi(cx_1 + x_2, y) = c\varphi(x_1, y) + \varphi(x_2, y)$.

Definition 1.3. An inner product or metric $\langle \cdot, \cdot \rangle$ on a vector space $V$ is a symmetric bilinear form.

1. **Positive definite** if $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
2. **Non-degenerate** for all $x \neq 0 \in V$, there exists $y \in V$ such that $\langle x, y \rangle \neq 0$.

Remark 1.4. Inner products may be represented as $\langle x, y \rangle$ or $\varphi(x, y)$ in this paper. For the purposes of this paper, it is assumed that $\langle \cdot, \cdot \rangle$ is a positive-definite inner product.
Definition 1.5. Let \( \varphi \) be a symmetric bilinear form. A canonical algebraic curvature tensor \( R_\varphi \) is an algebraic curvature such that, for all \( x, y, z, w \in V \),
\[
R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).
\]

Definition 1.6. A model space is defined as \( \mathcal{M} := (V, \varphi, R) \), where \( V \) is a vector space, \( \varphi \) is a metric, and \( R \) is an algebraic curvature tensor.

Definition 1.7. The structure group of a model space is, for all \( x, y, z, w \in V \),
\[
G_\mathcal{M} = \{ A \in GL(V) : R(x, y, z, w) = R(Ax, Ay, Az, Aw) \text{ and } \varphi(x, y) = \varphi(Ax, Ay) \}\}

Definition 1.8. Given a model space, the sectional curvature \( \kappa \) of a 2-plane \( \pi = \text{span}\{x, y\} \) is a function that inputs a 2-plane and outputs a real value, defined by
\[
\kappa(\pi) = \kappa(\langle x, y \rangle) = \frac{R(x, y, y, x)}{(x,x)\langle y,y \rangle - (x,y)^2}.
\]
Sectional curvature is independent of the basis used. Thus if \( \pi = \text{span}\{v, w\} \) rather than \( \text{span}\{x, y\} \), then \( \kappa(\langle x, y \rangle) = \kappa(\langle v, w \rangle) \).

Definition 1.9. An invariant subspace of a model space is a subspace \( W \) of the vector space \( V \) such that for any element \( A \in G_\mathcal{M} \), \( W \) will map to itself. It may also be expressed as:
If \( W \) is an invariant subspace, where \( W \subseteq V \), then for any \( A \in G_\mathcal{M} \), \( A : W \to W \).

Definition 1.10. If \( W \) is a subspace of \( V \), then \( W^\perp = \{ v \in V : v^\perp \subseteq W \} \). \( W^\perp \) is referred to as the perp space of \( W \).

2. **General \( G_\mathcal{M} \) Facts**

Some drawbacks about the algebraic curvature tensor \( R \) is that the value may depend on the basis used to reference \( V \). We take the following example to illustrate.

**Example 2.1.** Suppose we have a 3-dimensional vector space \( V \) with an orthonormal basis \( \{e_1, e_2, e_3\} \), and suppose that the following values of \( R \) are defined as, up to the curvature symmetries:
\[
R_{1223} = 2,
R_{2332} = 3,
\text{anything else} = 0.
\]

However, if we consider another basis \( \{f_1, f_2, f_3\} \), where \( f_1 = e_1 + e_2, f_2 = e_2, \) and \( f_3 = e_3 \). Then if we were to calculate \( R(f_1, f_2, f_2, f_3) \), then
\[
R(f_1, f_2, f_2, f_3) = R(e_1 + e_2, e_2, e_2, e_3),
= R(e_1, e_2, e_2, e_3) + R(e_3, e_2, e_2, e_3),
= R_{1223} + R_{2332},
= 2 + 3,
= 5.
\]
We could write the change of basis of \( \{ e_i \}_{i=1}^3 \) to \( \{ f_i \}_{i=1}^3 \) as a linear transformation, or matrix, with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

and declare that this matrix \( A \) cannot be in the structure group. Thus we can see how the structure group may need to come into play. If we take the collection of matrices that preserve the algebraic curvature tensor \( R \), and the inner product \( \varphi \), what would they look like?

This leads us to what we know about invariant subspaces. Given that we know about matrices, we want to find out the spaces that map to themselves, without trying to find the matrices blindly. The following example illustrates what we may gather from invariant subspaces.

**Example 2.2.** If we have a 5-dimensional vector space \( V \) with the orthonormal basis \( \{ e_1, e_2, e_3, e_4, e_5 \} \), and we know that \( W = \text{span} \{ e_1, e_2 \} \) is an invariant subspace of \( V \), then it is necessarily true that any \( A \in G_{35} \) must look like

\[
A = \begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 \\
a_2 & b_2 & c_2 & d_2 & e_2 \\
0 & 0 & c_3 & d_3 & f_3 \\
0 & 0 & c_4 & d_4 & f_4 \\
0 & 0 & c_5 & d_5 & f_5 \\
\end{bmatrix}
\]

since \( A \) must send \( W \) to itself.

The upper-right corner of the matrix still has elements in terms of \( e_1 \) and \( e_2 \), however. But, what is known about perp spaces will come to be convenient for reducing that space even more.

*It is known that when we have a non-degenerate vector space \( V \), and an invariant subspace \( W \), that \( W^\perp \) is also an invariant subspace.*

This is evident from the following theorem.

**Theorem 2.3.** If \( W \) is an invariant subspace of \( V \), and \( w \in W^\perp \), then \( Aw \in W^\perp \), for \( A \in G_{35} \).

**Proof.** Let \( v \in W \) such that \( W \) is an invariant subspace of \( V \). Let \( w \in W^\perp \) and let \( A \in G_{35} \).

Since \( W \) is an invariant subspace, and \( v \in W \), then \( Av \in W \), which implies \( A^{-1}v \in W \). Recall \( \varphi(v, w) = 0 \) by the definition of \( W^\perp \). We take \( Aw \), and find that

\[
\varphi(v, Aw) = A\varphi(A^{-1}v, w),
\]

since \( A^{-1}v \in W \), then

\[
A\varphi(A^{-1}v, w) = 0,
\]

\[\Rightarrow \quad \varphi(v, Aw) = 0,
\]

\[\Rightarrow \quad Aw \in W^\perp.
\]

\[\square\]
Thus in example 2.2, we can further reduce $A$ to

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & d_3 & f_3 \\ 0 & 0 & c_4 & d_4 & f_4 \\ 0 & 0 & c_5 & d_5 & f_5 \end{bmatrix}.$$ 

Meanwhile, we also want to understand what this means for the structure group $G_{3\mathfrak{m}}$. We see that if we have an invariant subspace in $V$, then our structure group elements become more defined. It is known according to [2] that the $\ker R$ is also an invariant subspace. Some studies use this fact to aid in their research, however that will not be the case in this paper, although it is important to mention regardless.

3. Sectional Curvature

So how exactly can we determine invariant subspaces then? This is what this research paper will focus on, and we begin with some facts from [3].

**Lemma 3.1.** Let $\varphi \in S^2(V^*)$, and suppose $R_\varphi \neq 0$. If $\pi$ is a 2-plane whose sectional curvature is extremal, then there exists an orthonormal basis of eigenvectors for $\pi$.

**Theorem 3.2** (Sectional Curvature is Bounded). Let $\varphi \in S^2(V^*)$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\varphi$, repeated according to multiplicity. Let $m$ and $M$ be the minimum and maximum of the set $\{\lambda_i \lambda_j | i \neq j\}$, respectively. The set of sectional curvatures of $R_\varphi$ is exactly the interval $[m, M]$.

We can come up with an example to motivate the uses of sectional curvature. One things we can do are constructing a basis which is inspired by geometry. Suppose we have a model space where $\dim V = 3$ and $V$ has a unique maximal and minimal sectional curvature, made by two distinct 2-planes, called $\pi_m$ and $\pi_M$ respectively. When we visualize these planes, they must intersect at some line of intersection, call this line $\pi_{m \cap M}$. Because $\pi_{m \cap M}$ is 1-dimensional, then we can span that line with a vector $v_{m \cap M}$. This vector will be a basis vector for $V$, and we can obtain two other basis vectors for $V$ by taking a vector to span $\pi_{m \cap M}$ and $\pi_M$, which each of the previous spaces must be 1-dimensional. And behold, we now have some specific method to constructing a basis that (although nonchalant) can aid in one’s studies.

This was only to illustrate how to construct a basis which, until further research shows, may or may not be useful, but at least is some consistent basis for $\dim V = 3$ with extremal sectional curvature.

4. Results

**Lemma 4.1** (The Structure Group Preserves Sectional Curvature). If $\mathfrak{M}$ is the model space $(V, \langle \cdot, \cdot \rangle, R)$ and $V$ has a 2-plane $\pi$, then for any $A \in G_{3\mathfrak{m}}$, $\kappa(A\pi) = \kappa(\pi)$.

**Proof.** Suppose that $\pi = \text{span}\{v, w\}$ is a 2-plane. Then
Given $A \in G_{2\mathfrak{M}}$, then

$$\kappa(A) = \kappa(Av, Aw) = \frac{R(Av, Aw, Av)}{\langle Av, Av \rangle \langle Aw, Aw \rangle - \langle Av, Aw \rangle^2}.$$ 

$$\kappa(A) = \frac{R(v, w, v)}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2}.$$ 

$$\Rightarrow \quad \kappa(A) = \kappa.$$ 

\[\square\]

**Theorem 4.2** (Unique Minimal and Maximal 2-planes are Invariant). Suppose $2\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, R)$ and that there is a unique 2-plane $\pi$ such that $\kappa(\pi)$ is an extrema. Then $A : \pi \to \pi$, where $A \in G_{2\mathfrak{M}}$.

**Proof.** If we have only one 2-plane $\pi$ such that $\kappa(\pi) = m$ or $M$, and that $A \in G_{2\mathfrak{M}}$ preserves the sectional curvature, then $A$ must map $\pi$ to itself.

The consequences of this theorem are significant. If it happens to be that we have a model space with one 2-plane for each of the bounds, then we know then that we have an invariant subspace. It may be more obvious so in this example.

**Example 4.3** (A Finite Structure Group). Suppose we have a model space $2\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, R_\varphi)$ where $\dim V = 3$, and $\{e_1, e_2, e_3\}$ is an orthonormal basis for $V$, and that $\langle e_i, e_j \rangle = \delta_{ij}$, and that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

We can identify the 2-planes which bound the sectional curvature $\kappa$. Here, we can identify that the set of eigenvalues of $\varphi$ are $\{1, 2, 3\}$. According to [3], the set of sectional curvatures of $R_\varphi$ are bounded as $[2, 6]$. The 2-plane $\pi_m$ where $\kappa(\pi_m) = 2$ must be spanned by the vectors $\{e_1, e_2\}$ since, by [3], and $\pi_M$ where $\kappa(\pi_M) = 6$ is similarly spanned by $\{e_2, e_3\}$.

According to the Theorem 4.2, then when we consider $A \in G_{2\mathfrak{M}}$, then we must preserve the sectional curvature, and map toward ourselves if there is only one 2-plane to produce an extrema.

If we consider the 2-plane $\pi_m$, then $\kappa(\pi_m) = \kappa(A\pi_m) = 2$. Thus $A\pi_m = \pi_m$. If we replace the $\pi_m$ with the vectors that span the plane, $\{e_1, e_2\}$, then we see that $A$ must leave that space invariant. Since $\{e_1, e_2\}$ is an invariant subspace, it it also true that $\{e_1, e_2\}^\perp = \{e_3\}$ is also an invariant subspace. Thus,

$$A = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}.$$
Recall though that since we also have to consider the 2-plane which produces maximal sectional curvature, that $A$ will further be restricted. Thus when $\pi_M$ was spanned by \{e_2, e_3\}, then we have another invariant subspace, where also the perp space is an invariant. $A$ is now restricted to the form

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}. $$

Thus when we operate $A$ with any basis vector $e_i$, then we have $Ae_1 = a_1e_1, Ae_2 = b_2e_2,$ and $Ae_3 = c_3e_3$. If we recall that $\langle e_i, e_j \rangle = \delta_{ij}$, then we see that for $e_1$ that

$$1 = \langle e_1, e_1 \rangle,
= \langle Ae_1, Ae_1 \rangle,
= \langle a_1e_1, a_1e_1 \rangle,
= (a_1)^2 \langle e_1, e_1 \rangle,
= (a_1)^2,
\implies a_1 = \pm 1.$$

We repeat for $e_2$ and $e_3$. Thus $a_1, b_2, c_3 = \pm 1$. Therefore, of all combinations, we see here that we have a structure group of 8 elements. Listed, they are

$$G_{\mathfrak{M}} = \{ [1 0 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [-1 0 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [-1 0 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, [1 0 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, [-1 0 0] \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \}. $$

In fact, we have covered the structure groups for which the $\varphi$ matrix is diagonal and has unique extremal 2-plane sectional curvature.

5. OPEN QUESTIONS

- Using the minimum and maximum sectional curvatures for 3-dimensional cases to be solved with ease. Is there a way this method can be generalized for larger dimensional cases?
- What happens when we have multiple 2-planes which provide an extremal sectional curvature?
- In the case where $R_{r} = R_{\varphi} + R_{\psi}$, where $\varphi$ is positive definite, is it true that when $R_{r} = R_{\varphi} + R_{\psi}$ that $\psi = \pm \psi^T$?
- In the model spaces of varying signatures, what can we observe when we have at least one extremal 2-plane spanned by 2 vectors $e_i$ and $e_j$ where $\langle e_i, e_j \rangle = -\langle e_i, e_j \rangle$?
- If we investigate the kernels of the model spaces, we also have an invariant subspace, discovered from [2]. Is there a way to connect this invariant subspace with our method of finding invariant subspaces?
- Let $\alpha_i$ be a tensor of any type. If we have a model space $\mathfrak{M}_1 = (V, \alpha_1)$ and find the structure group $G_{\mathfrak{M}_1}$, and for the model space $\mathfrak{M}_2 = (V, \alpha_1, \alpha_2)$, what does the structure group look like for $G_{\mathfrak{M}_2}$? What does $G_{\mathfrak{M}_1}$ look like for $\mathfrak{M}_1$?

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