LINEAR DEPENDENCE OF CANONICAL ALGEBRAIC CURVATURE TENSORS WITH ASSOCIATED CHAIN COMPLEXES

DAVID WILLIAMS

August 20, 2015

ABSTRACT

Our goal is to develop chain complexes as a tool to analyze linearly dependent sets of canonical algebraic curvature tensors built from degenerate operators. We provide an example chain complex and outline our methods for the study of complexes alongside linear dependence equations. We derive some basic computational results from relationships that are common in working with chain complexes. Later, we lay groundwork for the categorization of chain complexes and list projects for future study.
1 Introduction

1.1 Canonical Algebraic Curvature Tensors

**Definition 1.** Let $V$ be a real, finite-dimensional vector space. Let $R : V^4 \to \mathbb{R}$ be a multilinear function. Then, we call $R$ an **algebraic curvature tensor** if it satisfies the following properties:

1. $R(x, y, z, w) = -R(y, x, z, w)$
2. $R(x, y, z, w) = R(z, w, x, y)$
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

We call the last property the Bianchi Identity. We denote the vector space of all algebraic curvature tensors on $V$ as $A(V)$.

In an inner product space, there is a natural way to construct an algebraic curvature tensor from the inner product.

**Theorem 2.** Let $V$ be a vector space with positive-definite inner product $\phi$. Then,

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

is an algebraic curvature tensor.

We can extend this construction to certain linear operators and matrices on $V$ as follows.

**Definition 3.** Let $V$ be a vector space with positive-definite inner product $\phi$. Let $\tau$ be a symmetric bilinear form on $V$. Let $A : V \to V$ be the self-adjoint linear operator characterized by $\tau(x, y) = \phi(Ax, y)$ for all $x, y \in V$. Then, we say that the following are **canonical algebraic curvature tensors of symmetric build**:

$$R_A^S(x, y, z, w) = R_\phi(Ax, Ay, z, w) = \phi(Ax, w)\phi(Ay, z) - \phi(Ax, z)\phi(Ay, w)$$

$$R_\tau^S(x, y, z, w) = \tau(x, w)\tau(y, z) - \tau(x, z)\tau(y, w)$$

$$\implies R_A^S, R_\tau^S \in A(V)$$

Now, let $\psi$ be an antisymmetric bilinear form on $V$, and let $B : V \to V$ be the skew-adjoint linear operator characterized by $\psi(x, y) = \phi(Bx, y)$. We say that the following are **canonical algebraic curvature tensors of anti-symmetric build**:

$$R_B^A(x, y, z, w) = \phi(Bx, w)\phi(By, z) - \phi(Bx, z)\phi(By, w) - 2\phi(Bx, y)\phi(Bz, w)$$

$$R_\psi^A(x, y, z, w) = \psi(x, w)\psi(y, z) - \psi(x, z)\psi(y, w) - 2\psi(x, y)\psi(z, w)$$

$$\implies R_B^A, R_\psi^A \in A(V)$$
This idea of building an algebraic curvature tensor out of a matrix or bilinear form is central to our study the tensors themselves. Throughout this paper, we will say two canonical algebraic curvature tensors $R_A$ and $R_B$ have the “same build” if $A$ and $B$ are either both symmetric or both antisymmetric.

By [2] for $\text{Rank}(A), \text{Rank}(\tau) \geq 3$, we have that $R^S_A, R^S_\tau \in \mathcal{A}(V) \iff A = A^*$ and $\tau$ is a symmetric form. Similarly for $\text{Rank}(B), \text{Rank}(\psi) \geq 3$, we have that $R^A_B, R^A_\psi \in \mathcal{A}(V) \iff B = -B^*$ and $\tau$ is an anti-symmetric form. This justifies our use of the phrase “algebraic curvature tensor” when referring to tensors built from matrices and operators.

We wish to consider when a set of canonical algebraic curvature tensors (of either build) is linearly independent. When the build of a particular tensor is unimportant or unknown, we may omit the superscript: $R_A$ or $R_\tau$. Furthermore, we will assume throughout that $V$ denotes a real, finite-dimensional vector space equipped with a positive-definite inner product $\phi$, so that $M$ is a symmetric operator on $V$ if and only if $M = M^*$, and $M$ is an antisymmetric operator on $V$ if and only if $M = -M^*$.

The multilinearity of a canonical algebraic curvature tensor allows us to simplify the operators out of which they are built. If $A$ is a matrix over $V$ or an operator on $V$ and $c \in \mathbb{R}$ is nonzero, then $R_{cA}(x, y, z, w) = R_\phi(cAx, cAy, z, w) = c^2 R_\phi(Ax, Ay, z, w) = c^2 R_A(x, y, z, w)$. Thus, scalars multiplying a canonical algebraic curvature tensor can be “absorbed” into the subscript up to a sign. Thus, when we are considering an equation of the form

$$\sum_{i=0}^{k} \alpha_i R_{A_i} = 0$$

we will assume that $\alpha_i \in \{1, -1\}$.

To work with canonical algebraic curvature tensors, we will need to talk about the kernel of an algebraic curvature tensor.

**Definition 4.** Let $R : V^4 \to \mathbb{R}$ be an algebraic curvature tensor. We define the kernel of $R$ as

$$\text{Ker} R = \{ x \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V \}$$

This definition may seem “biased” toward the first slot. However, Dunn, Franks and Palmer [7] have shown that

$$\text{Ker} R = \{ y \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V \}$$

$$= \{ z \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V \}$$

$$= \{ w \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V \},$$

so we need not worry that the different slots of $R$ have different kernels.

In discussing linear independence, we wish to focus our study on cases in which linear dependence is nontrivial. To that end, we establish the idea of proper linear dependence.

**Definition 5.** Let $S = \{Q_1, ..., Q_r\}$ be a linearly dependent set. We say that $S$ is properly linearly dependent if $A$ is a linearly independent set for all proper $A \subset S$. 

3


1.2 Chain Complexes

To place restrictions on our different sets of tensors, we borrow an idea from algebraic topology: the chain complex. For our purposes, we will consider only complexes of operators that all act on the same vector space.

**Definition 6.** Let $V$ be a vector space, and let $A_1, ..., A_k : V \to V$ be linear operators on $V$. If $\text{Im} A_i \subset \text{Ker} A_{i+1}$ for $1 \leq i \leq k - 1$, then we call $D = (A_1, ..., A_k)$ a **chain complex**, and we write

$\begin{array}{cccc}
A_1 & A_i & A_{i+1} & A_k \\
V & \to & \to & \to \\
\end{array}$

If $D_1, ..., D_m$ are chain complexes with $D_i = (A^i_1, ..., A^i_{k_i})$, we call $E = \cup_{i=1}^m D_i$ a **compound chain complex** and we write

$\begin{array}{cccc}
A^1_1 & A^1_j & A^1_{j+1} & A^1_{k_1} \\
V & \to & \to & \to \\
A^2_1 & A^2_j & A^2_{j+1} & A^2_{k_2} \\
V & \to & \to & \to \\
& & & \\
& & & \\
A^m_1 & A^m_j & A^m_{j+1} & A^m_{k_m} \\
V & \to & \to & \to \\
\end{array}$

We will study linearly dependent sets of algebraic curvature tensors constructed from a set of operators which satisfy a compound chain complex. Throughout, we will assume that every operator in the complex is represented in the associated linear dependence equation and vice versa. Studying the operator from which an algebraic curvature tensor was built can give us information about the tensor itself, by the following result from Gilkey:

**Theorem 7.** [3] Let $V$ be a real, finite-dimensional vector space, and let $A : V \to V$ be an operator on $V$. Then,

1. If $\text{Rk} A \leq 1$, then $R_A = 0$, and
2. If $\text{Rk} A \geq 2$, then $\text{Ker} R_A = \text{Ker} A$.

Thus, a chain complex structure also gives us information about the algebraic curvature tensors constructed from the operators. Depending on the assumptions involved, we may be able to put bounds on the dimension of the vector space as well.

For a given compound chain complex with a linear dependence equation, We wish to combine the information derived from a the complex with our knowledge about tensor behavior. To that end, we introduce an operation which allows for some convenient results.

4
1.3 Precomposition

**Definition 8.** Let $A$, $B$ be operators on a real, finite-dimensional vector space $V$, and let $R_A$ be the canonical algebraic curvature tensor built from $A$. Then, we define pre-composition by $B$, denoted $B^*R_A$, by the following

$$B^*R_A(x, y, z, w) = R_A(Bx, By, Bz, Bw)$$

If $A = A^*$, the above construction yields

$$R^S_A(Bx, By, Bz, Bw) = \phi(ABx, Bw)\phi(ABy, Bz) - \phi(ABx, Bz)\phi(ABy, Bw)$$

$$= \phi(B^*ABx, w)\phi(B^*ABy, z) - \phi(B^*ABx, z)\phi(B^*ABy, w)$$

$$= R_\phi(B^*ABx, B^*ABy, z, w)$$

$$= R^S_{BA^*} = R^S_{BAB}$$

Remarkably, it is possible to derive the same result if $A = -A^*$. However, we first need a key theorem from McMahon:

**Theorem 9.** [5] Let $B$ be an operator on $V$ with $A = A^*$ with respect to $\phi$. Then,

$$R^A_A(x, y, z, w) = 2R^S_A(x, y, z, w) + R^S_A(x, z, y, w) + R^S_A(x, w, z, y)$$

$$= 2R^S_\phi(Ax, Ay, z, w) + R^S_\phi(Ax, Az, y, w) + R^S_\phi(Ax, Aw, z, y).$$

If $A = -A^*$, then we have

$$B^*R^A_A = 2R^S_\phi(ABx, ABy, Bz, Bw) + R^S_\phi(ABx, ABz, By, Bw) + R^S_\phi(ABx, ABw, Bz, By)$$

$$= 2R^S_\phi(B^*ABx, B^*ABy, z, w) + R^S_\phi(B^*ABx, B^*ABz, y, w) + R^S_\phi(B^*ABx, B^*ABw, z, y)$$

$$= 2R^S_{BA^*}(x, y, z, w) + R^S_{BA^*}(x, z, y, w) + R^S_{BA^*}(x, w, z, y)$$

$$= R^A_{BA^*}(x, y, z, w) = R^A_{BAB} = R^A_{BAB}$$

by McMahon’s identity above.

It follows readily from this construction that for symmetric or anti-symmetric operators $A$ and $B$, if $BA = 0$, then $A^*R_B = B^*R_A = 0$. In working with chain complexes and associated linear dependence equations, it is often the case that a tensor will vanish under precomposition by a certain operator. We make use of this result to reduce the number of terms in our linear dependence equation.

## 2 Previous Work

The association of a chain complex to a linear dependence equation for algebraic curvature tensors was previously studied in [5]. Notable results are reproduced here without proof, which can be found there.
Lemma 10. [5] If \(A, B\) are symmetric or antisymmetric operators with \(\text{Im} B \subset \text{Ker} A\) or \(\text{Im} A \subset \text{Ker} B\), then

\[B^* R_A = R_{B^* A} = R_{\pm BAB} = R_{BAB} = 0\]

This is a corollary to Theorem 7, and the connecting bridge between linear dependence of algebraic curvature tensors and compound chain complexes.

Theorem 11. Let \(A, B, C, D\) be operators on \(V\) with \(\text{Rk} A, \text{Rk} B, \text{Rk} C, \text{Rk} D \geq 4\). Suppose \(\alpha_1 R_B + \alpha_2 R_C + \alpha_3 R_D = 0\) and the operators satisfy the chain complex

\[
\begin{array}{ccc}
V & \xrightarrow{A} & V \\
\downarrow{D} & & \downarrow{B} \\
V & \xleftarrow{C} & V
\end{array}
\]

Then, we have

1. \(R_A\) and \(R_C\) have the same build, and \(A^3 C = \pm ACAC\) and \(AC^3 = \pm ACAC\), and
2. \(R_B\) and \(R_D\) have the same build, and \(B^3 D = \pm BDBD\) and \(BD^3 = \pm BDBD\).

Theorem 12. Let \(A, B_1, ..., B_k\) be operators on vector space \(V\) such that \(0 = R_A + \sum \alpha_i R_{B_i}\). Suppose the operators fit one of the following compound chain complexes:

\[
\begin{array}{ccc}
A & \xrightarrow{B_i} & V \\
V & \rightarrow & V \\
\end{array}
\]

, or

\[
\begin{array}{ccc}
B_i & \xrightarrow{A} & V \\
V & \rightarrow & V \\
\end{array}
\]

For symmetric or antisymmetric \(A\), \(R_A = 0\). For antisymmetric \(A\), if the sequence is exact for some \(B_i\), then \(B_i\) is invertible.

These results were achieved through precomposition by the operator \(A\) and use of Lemma 10. Following are some other previous results that we will reference.

Lemma 13. If \(A = \pm A^*\) is an operator on \(V\) and \(p, k \in \mathbb{N}\), then

\[\text{Rk} A = p \iff \text{Rk} A^k = p\]
Proof. The backwards direction was proven in [5]. To show the forward direction, first assume $A$ is symmetric. Diagonalize $A$, so that $A^k$ is diagonalized as well. Then $RkA$ is the number of nonzero diagonal entries of $A$ and $RkA^k$ is the number of nonzero diagonal entries of $A^k$. So if $A = [a_{ij}]$ and $A^k = [\hat{a}_{ij}]$, then for all $i$,

$$\hat{a}_{ii} = a_{ii}^k$$

So, $\hat{a}_{ii} = a_{ii}^k = 0 \iff a_{ii} = 0$. Therefore, the number of nonzero diagonal entries of $A^k$ is exactly the same as the number of nonzero diagonal entries of $A$.

For $A$ antisymmetric, block-diagonalize $A$ with 2-by-2 blocks on the diagonal and 0 elsewhere. Then, for any nonzero block $\hat{A}$ of $A$, we have

$$\hat{A} = \lambda M,$$

where

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $\lambda \neq 0$. Since $A$ is block-diagonal, each $\lambda M$ block in $A$ corresponds to exactly one $\lambda^p M^p$ block in $A^p$. Since $\lambda \neq 0$ and $M$ is idempotent, $\lambda^p M^p \neq 0$. Thus, if there are $\frac{p}{2}$ nonzero blocks in $A$, there must be $\frac{p}{2}$ nonzero blocks in $A^p$. Therefore, $\text{Rank}(A) = \text{Rank}(A^p)$. \qed

Lemma 14. [3] If $A, B$ are operators on $V$ with $R_A = R_B$, then $A = \pm B$ if

1. $A, B$ are symmetric and $RkA \geq 3$, or
2. $A, B$ are antisymmetric.

Lemma 15. [1] [6] If $A, B$ are operators on $V$ with $RkA \geq 3$,

1. if $A, B$ are symmetric, then $R^S_A \neq -R^S_B$, and
2. if $A, B$ are antisymmetric, then $R^\Lambda_A \neq -R^\Lambda_B$.

Lemma 16. [6] [4] Let $A$ be an antisymmetric operator on $V$ with $RkA \geq 4$. If $B$ is a symmetric operator on $V$, then

$$R^\Lambda_A \neq \pm R^S_B.$$  

For convenience, we combine the previous three lemmas for the following result:

Corollary 17. Suppose $A, B$ are symmetric or antisymmetric operators on $V$ with $RkA \geq 3$ and we have

$$R_A = \pm R_B.$$  

Then, $R_A$ and $R_B$ have the same build, $R_A = R_B$, and $A = \pm B$.
3 Motivation

Much of my work has been generalizations on results I found working with a particular chain complex. A short summary of my findings with that complex follows.

**Example.** Let $A, B, C, D$ each be symmetric or antisymmetric operators on $V$ with $\text{Rank}(A), \text{Rank}(B), \text{Rank}(C), \text{Rank}(D) \geq 3$. Suppose these operators satisfy the compound chain complex

\[
\begin{array}{ccc}
V & \xrightarrow{A} & V \\
& \xleftarrow{B} & \downarrow{C} \\
V &\xrightarrow{D} & V
\end{array}
\]

Together with this diagram, we associate the linear dependence relationship $R_A + \alpha_1 R_B + \alpha_2 R_C + \alpha_3 R_D = 0$ with $\alpha_i \in \{1, -1\}$. Then,

1. $\alpha_3 = -1$.
2. $\alpha_1 = -\alpha_2$.
3. If $A$ is invertible, then $C = 0$.
4. If $B$ is invertible, then $D = 0$.
5. If $C$ is invertible, then $A = D = 0$, and $R_C = \pm R_B$.
6. If $D$ is invertible, then $B = C = 0$, and $R_A = \pm R_D$.

**Proof.** Examination begins by precomposing this equation with each of $A, B, C, D$ in order to achieve four new linear dependence equations:

\[
\begin{align*}
R_{A^3} + \alpha_1 R_{ABA} + \alpha_3 R_{ADA} &= 0, \\
R_{BAB} + \alpha_1 R_{B^3} + \alpha_2 R_{BCB} &= 0, \\
\alpha_1 R_{C^3} + \alpha_2 R_{C^3} &= 0, \\
R_{DAD} + \alpha_3 R_{D^3} &= 0.
\end{align*}
\]

We omit tensors which are identically 0. Note that each of our new equations has fewer terms than our original equation. Intuitively, we have traded working with simple matrices and many tensors for working with more complicated matrices and fewer
tensors. This is the convenience of precomposition, and we are especially interested in cases where we are left with two-term equations, such as

\[ R_{C^3} = -\alpha_1 \alpha_2 R_{CBC}, \]
\[ R_{D^3} = -\alpha_3 R_{DAD}. \]

By Lemma 10 and Corollary 17, we have that \( \text{Rk} C^3, \text{Rk} D^3 \geq 3 \), and thus:

1. \( R_{D^3} \) and \( R_{DAD} \) have the same build and \( \alpha_3 = -1 \),
2. \( R_{C^3} \) and \( R_{CBC} \) have the same build and \( \alpha_1 = -\alpha_2 \).

With a result we will develop later, it can be shown further that \( R_A, R_D \) must have the same build and \( R_B, R_C \) must have the same build. Additionally, we note the following about the complex

1. If \( A \) is invertible, then \( V = \text{Im}(A) \subseteq \text{Ker}(C) \implies C = 0 \).
2. If \( B \) is invertible, then \( V = \text{Im}(B) \subseteq \text{Ker}(D) \implies D = 0 \).
3. If \( C \) is invertible, then \( V = \text{Im}(C) \subseteq \text{Ker}(D) \implies D = 0 \) and \( \text{Im}(A) \subseteq \text{Ker}(C) = 0 \implies A = 0 \). Also, the linear dependence equation reduces to \( R_B = \pm R_C \implies B = \pm C \) if \( \text{Rank}(B) \geq 3 \) or \( \text{Rank}(C) \geq 3 \).
4. If \( D \) is invertible, then \( \text{Im}(C) \subseteq \text{Ker}(D) = 0 \implies C = 0 \) and \( \text{Im}(B) \subseteq \text{Ker}(D) = 0 \implies B = 0 \). Also, the linear dependence equation reduces to \( R_A = \pm R_D \implies A = \pm D \) if \( \text{Rank}(A) \geq 3 \) or \( \text{Rank}(D) \geq 3 \).

We generalize this analysis to a compound chain complex \( \mathcal{D} \) relating \( k \) operators \( A_1, \ldots, A_k \) without making any assumptions about the underlying structure of the individual chain complexes in the compound.

1. Begin with a linear dependence equation of the form

\[ R_{A_1} + \alpha_2 R_{A_2} + \ldots + \alpha_k R_{A_k} = 0 \]

where the \( A_i \)'s are our operators and \( \alpha_i \in \{1, -1\} \) for \( 2 \leq i \leq k \).
2. Assume proper linear dependence of the canonical algebraic curvature tensors. In other words, assume that \( \text{Rank}(A_i) \geq 2 \) and \( \alpha_i \neq 0 \) for all \( i \).
3. Precompose this equation with each \( A_i \) separately to achieve a system of \( k \) linear dependence equations with fewer terms.
4. Do any of the equations have 2 or fewer terms? If so, refer to Corollary 17. With reasonable assumptions about operator ranks, this should allow us to solve for some of the \( \alpha_i \)'s.
5. Determine how many subsets of the operators have kernels which intersect nontrivially. When considering vectors from one of these intersections, several terms in the linear dependence equation necessarily vanish, giving us more concrete information about the remaining terms.

From this examination, some natural questions arise:

- How much information about operators does a chain complex encode?
- For a linear dependence equation with a chain complex, does reducing the number of terms by precomposition yield more information about the operators?
- For a given number of operators, how many chain complexes are possible?

In this paper, I will begin to answer all of these questions. My hope is to lay a strong foundation for the study of chain complexes with linear dependence equations of algebraic curvature tensors, and to motivate deeper study in the field.

4. Compound Chain Complexes and Their Operators

We want to study compound chain complexes in general, especially with respect to operators. Given a chain complex, we want to decide what kinds of operators could satisfy a dependence relation. We take a kernel-based approach, so that an operator splits the vector space \( V \) into two parts: the image of \( A \) and the kernel of \( A \). The following result justifies this perspective.

**Lemma 18.** Let \( A = \pm A^* \) be an operator on \( V \). Then,

\[
\text{Im}A \cap \text{Ker}A = \{0\}
\]

**Proof.** Suppose we have a nonzero \( v \in \text{Im}A \cap \text{Ker}A \). Let \( B = \{v, e_1, \ldots, e_{m-1}\} \) be a basis for \( \text{Im}A \). Then,

\[
\{Av, Ae_1, \ldots, Ae_{m-1}\}
\]

is a spanning set for \( \text{Im}A^2 \). But \( Av = 0 \) by assumption, so \( B' = \{Ae_1, \ldots, Ae_{m-1}\} \) is also a spanning set for \( \text{im}A^2 \). Thus,

\[
\text{Rk}A^2 = \dim(\text{Im}A^2) < \dim(\text{Im}A) = \text{Rk}A
\]

But this is a contradiction to Lemma 13, so no such \( v \) exists. \( \square \)

We now develop a foundational result for the study of chain complexes with associated linear dependence equations. Given an arbitrary chain complex and a set of operators (or characteristics which determine operators) satisfying that complex, we are unable to determine anything substantial about the underlying vector space. It seems possible that almost any set of degenerate operators could be made to satisfy a chain complex by allowing each operator to have an arbitrarily large kernel. This amounts to
fitting the operators into a vector space of arbitrarily large dimension. Thus, if we want to utilize chain complexes to study operators, we must be able to put boundaries on the dimension of the base vector space. We now develop a theorem which guarantees that such a boundary exists and produces the vector space of smallest valid dimension for the chain complex in question.

**Theorem 19.** Let \( \mathbb{D} = (A_1, \ldots, A_k) \) be a chain complex relating \( k \) operators on an \( n \)-dimensional vector space \( V \). Suppose that \( A_i = \pm A_i^* \) for \( 1 \leq i \leq k \), and let \( U = \cap_{i=1}^k \text{Ker} A_i \) be nontrivial. Then, there exists \( \overline{\mathbb{D}} = (\overline{A}_1, \ldots, \overline{A}_k) \), a chain complex on \( \overline{V} = V/U \) \( \text{Im} A_i \subseteq \text{Im} \overline{A}_j \) if \( \text{Im} A_i \subseteq \text{Ker} A_j \). \( \overline{\mathbb{D}} \) also has the property that that 

\[ \overline{U} = \cap_{i=1}^k \text{Ker} \overline{A}_i = \{0\} \]

\[ 
\begin{array}{ccc}
  V & \xrightarrow{A} & V \\
  \downarrow & & \downarrow \\
  V & \xrightarrow{B} & V \\
  \downarrow & & \downarrow \\
  V & \xrightarrow{C} & V \\
  \downarrow & & \downarrow \\
  V & \xrightarrow{D} & V \\
  \downarrow & & \downarrow \\
  V & \xrightarrow{E} & V \\
  \downarrow & & \downarrow \\
  V & \xrightarrow{F} & V \\
\end{array}
\]

**Proof.** Suppose there exists \( v \in U \) such that \( v \neq 0 \). Define

\[ \overline{V} = V/U, \]

\[ \pi : V \to \overline{V} \text{ given by } \pi(v) = v + U \text{ for all } v \in V, \]

\[ \pi^* \overline{R} = R \text{ so that } \pi^* \overline{R}(x, y, z, w) = R(x, y, z, w), \]

\[ \overline{A}_i : \overline{V} \to \overline{V} \text{ such that } \pi(A_i v) = \overline{A}_i(v + U) \text{ for all } v \in V, \]

We need to verify that the above is well-defined and simplifies our chain complex so that \( \overline{U} = \cap_{i=1}^k \text{Ker} \overline{A}_i = \{0\} \).

By Theorem 7 for \( R_A \neq 0 \), \( \text{Ker} R_A = \text{Ker} A \), and so \( U \subset \text{Ker} R_A \) for all \( i \). Let \( x_1, x_2 \in V \) such that \( x_1 + U = x_2 + U \). Then,
\[ x_1 - x_2 \in U \implies R_A(x_1 - x_2, y, z, w) = 0 \]
\[ \implies R_A(x_1, y, z, w) = R_A(x_2, y, z, w) \]
\[ \implies R_A(x_1 + U, y + U, z + U, w + U) = R_A(x_2 + U, y + U, z + U, w + U). \]

So, \( \bar{R} \) is well-defined for \( R \) on \( V \). Also, we have
\[ x_1 - x_2 \in U \subset \ker A_i \]
\[ \implies A_i(x_1 - x_2) = 0 \]
\[ \implies A_i(x_1) = A_i(x_2) \]
\[ \implies \pi(A_i x_1) = \pi(A_i x_2) \]
\[ \implies \bar{A}_i(x_1 + U) = \bar{A}_i(x_2 + U) \]

for all \( 0 \leq i \leq k \). So \( \bar{A}_i \) is well-defined.

We now need to show that \( \bar{D} \) is a chain complex with the same operator relationships as \( D \). Since the information encoded by a chain complex is a containment of the images of some operators in the kernels of others, it is sufficient to show that \( \text{Im} A_i \subset \text{Ker} \bar{A}_j \) for \( \text{Im} A_i \subset \text{Ker} A_j \). Fix such \( i, j \) and let \( v + U \in \text{im} A_i \) so that \( \bar{A}_i(u + U) = v + U \). Then,
\[ v - A_i u \in U \subset \text{Ker} A_j \]
\[ \implies A_j(v - A_i u) = 0 \]
\[ \implies A_j v = A_j A_i u = 0 \text{ since } \text{Im} A_i \subset \text{Ker} A_j \]
\[ \implies \bar{A}_j(v + U) = \pi(A_j v) = \pi(0) = 0 + U. \]

Thus, our new operators satisfy the same chain complex on \( \bar{V} \) that our original operators satisfied on \( V \). We also have that if \( v + U \in \bigcap_{i=0}^{k} \text{Ker} A_i \), then
\[ v + U \in \bigcap_{i=0}^{k} \text{Ker} \pi(A_i) \]
\[ \implies \pi(A_i v) = 0 + U \text{ for } i \leq k \]
\[ \implies A_i v \in U \subset \text{Ker} A_i \text{ for } i \leq k \]
\[ \implies A_i v \in \text{Ker} A_i \cap \text{Im} A_i = \{0\} \text{ for } i \leq k \]
\[ \implies v \in U \implies \pi(v) = 0 + U \]
by Lemma 18. Finally, to guarantee that $Rk\tilde{A}_i = RkA_i$, let $v = A_iu$ be nonzero for some $i \leq k$. Since $ImA_i \cap U \subset ImA_i \cap KerA_i = \{0\}$ by Lemma 18, $v \notin U$. Thus, $v + U = \pi(v) = \pi(A_iu) = \tilde{A}_i(u + U)$, and so $v + U \in Im(\tilde{A}_i)$.

Therefore, given a compound chain complex, we need only consider a vector space which is large enough to meet our assumptions about the operators, and no larger. We call a compound chain complex reduced if the intersection of the kernels of all its operators is trivial. This result is very convenient, because it allows us to assume without loss of generality for every chain complex that the intersection of the kernels of all operators is trivial.

4.1 Interaction of Kernels in a Chain Complex

Putting restrictions on $\dim V$ is generally a tedious process in which one must consider the interrelated kernels of all operators in the complex. However, assuming that the kernels have a trivial intersection gives us some leverage when considering how the kernels interact.

**Theorem 20.** Let $A_1, \ldots, A_k$ be operators on a vector space $V$ such that $\{R_{A_1}, \ldots, R_{A_k}\}$ is a properly linearly dependent set. Suppose that $\bigcap_{i=1}^{k} KerA_i = 0$, and let $I \subset \{1, \ldots, k\}$ be an index set with $|I| = k - 1$. Then,

$$\bigcap_{i \in I} KerA_i = \{0\}$$

*Proof.* Since $\{R_{A_1}, \ldots, R_{A_k}\}$ is a properly linearly dependent set, $R_{A_i} \neq 0 \implies \text{Rank}(A_i) \geq 2$ for all $i$ by Theorem 7. Thus, $KerR_{A_i} = KerA_i$ by, and so $\bigcap_{i=1}^{k} KerA_i = \{0\}$. Furthermore, we can choose $\alpha_1, \ldots, \alpha_k$ all nonzero such that

$$\sum_{i=1}^{k} \alpha_iR_{A_i} = 0$$

Now let $j$ be an index such that $\{j\} \cup I = \{1, \ldots, k\}$, and let $v \in A_j$. We can rewrite the linear dependence equation

$$-\alpha_jR_{A_j} = \sum_{i=1}^{j-1} \alpha_iR_{A_i} + \sum_{i=j+1}^{k} \alpha_iR_{A_i}$$

$$\implies R_{A_j} = -\frac{1}{\alpha_j} \left(\sum_{i=1}^{j-1} \alpha_iR_{A_i} + \sum_{i=j+1}^{k} \alpha_iR_{A_i}\right)$$

$$\implies R_{A_j}(v, y, z, w) = -\frac{1}{\alpha_j} \left(\sum_{i=1}^{j-1} \alpha_iR_{A_i}(v, y, z, w) + \sum_{i=j+1}^{k} \alpha_iR_{A_i}(v, y, z, w)\right)$$

$$= -\frac{1}{\alpha_j} \left(\sum_{i=1}^{j-1} 0 + \sum_{i=j+1}^{k} 0\right)$$

$$= 0$$

for all $y, z, w \in V$. Thus, $v \in KerR_{A_j} \implies v \in KerA_j$. But by choice of $v$, we now have $v \in \bigcap_{i=1}^{k} KerR_{A_i} = \{0\}$ by assumption. So $v = 0$. \[\square\]
5 Precomposed Linear Dependence Equations

The process of precomposition by a symmetric or antisymmetric operator is key to our study of chain complexes and the linear dependence of canonical algebraic curvature tensors. The equations that result from precomposition are often simpler at the tensor level, but the operators from which the tensors are built are often much more complicated. However, because we can often reduce our equations to 2 or 3 nontrivial terms, we can begin to apply previous knowledge to derive matrix equations.

The following result justifies our use of precomposition to form new canonical algebraic curvature tensors.

**Lemma 21.** Let $A = \alpha A^*$ and $B = \beta B^*$ for some $\alpha, \beta = 1, -1$. Then, we have

$$(ABA)^* = \beta ABA$$

**Proof.** By properties of adjoints, we have

$$(ABA)^* = A^* B^* A^* = (\alpha A)(\beta B)(\alpha A) = \alpha^2 \beta ABA = \beta ABA$$

since $\alpha^2 = 1$.

As a corollary to the above, precomposing a canonical algebraic curvature tensor by a symmetric or antisymmetric operator produces another canonical algebraic curvature tensor of the same build. This allows us to apply previous results to tensors that have undergone precomposition.

5.1 $R_{A^3} = \pm R_{ABA}$ and $R_{B^3} = \pm R_{BAB}$ for Symmetric $A, B$

The following result is based on a relationship between canonical algebraic curvature tensors that arises in certain chain complex structures.

**Theorem 22.** Let $A, B$ be operators on a vector space $V$ such that $A = A^*$, $B = \pm B^*$ and $3 \leq RkB \leq RkA$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Suppose that

$$R_{A^3} = \pm R_{ABA}$$
$$R_{B^3} = \pm R_{BAB}$$

We choose a basis for $V$ such that $A$ is diagonal. Then, we have that for distinct $i, j \leq \text{dim}V$,

$$a_{ii} = a_{jj}, \text{ or}$$
$$b_{ij} = b_{ji} = 0$$

**Proof.** By Corollary 17, we have that $B = B^*$, $R_{A^3} = R_{ABA}$ and $R_{B^3} = R_{BAB}$. Also,
\[ \beta A^3 = ABA \text{ for some } \beta = 1, -1 \]

\[ \eta B^3 = BAB \text{ for some } \eta = 1, -1 \]

\[ \implies \beta A^3 B = ABAB = \eta AB^3 \]

\[ \implies A^3 B = \gamma AB^3 \text{ for } \gamma = \beta \eta \]

Thus with diagonalized A, we have

\[
A = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
& \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 \\
\end{bmatrix}
\]

\[ \implies A^3 = \begin{bmatrix}
\lambda_1^3 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
& \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 \\
\end{bmatrix}
\]

where \( h = RkA \). Let \( B^3 = [\tilde{b}_{ij}] \). Thus, from the above relation, we have

\[
A^3 B = \begin{bmatrix}
\lambda_1^3 b_{11} & \lambda_1^3 b_{12} & \ldots & \lambda_1^3 b_{1h} & 0 & \ldots & 0 \\
\lambda_2^3 b_{21} & \lambda_2^3 b_{22} & \ldots & \lambda_2^3 b_{2h} & \ddots & \ddots & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
\lambda_h^3 b_{h1} & \ldots & \lambda_h^3 b_{hh} & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \lambda_h^3 b_{hh} & \ddots & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \ldots & \ldots & 0 & \ddots & \ddots & \ddots \\
\end{bmatrix} = [\lambda_i^3 b_{ij}]
\]
\[ AB^3 = \gamma [\lambda b_{ij}] \]

Since \( B, B^3 \) are symmetric matrices, we have that \( b_{ij} = b_{ji} \) and \( \tilde{b}_{ij} = \tilde{b}_{ji} \) for all \( i, j \).

Thus, for distinct \( i, j \) we have

\[ \lambda^3 b_{ij} = \gamma \lambda_i b_{ij} \quad \text{and} \quad \lambda^3 b_{ji} = \gamma \lambda_j b_{ji} \]

\[ \implies \lambda^2 b_{ij} = \gamma b_{ij} = \gamma b_{ji} = \lambda^2 b_{ji} \quad \text{since} \quad \lambda_k \neq 0 \quad \text{for all} \quad k \leq m \]

\[ \implies (\lambda_i^2 - \lambda_j^2) b_{ij} = 0 \]

\[ \implies (a_{ii}^2 - a_{jj}^2) b_{ij} = 0 \]

\[ \implies a_{ii} = \pm a_{jj} \quad \text{or} \quad b_{ij} = b_{ji} = 0 \quad \text{for distinct} \quad i, j \]

This result is very nearly sufficient to show commutativity for symmetric \( A, B \) given our assumptions. For diagonalized \( A \), we have \( AB = [a_{ii}b_{ij}] \) and \( BA = [b_{ija_{jj}}] \). Thus, in order for \( A, B \) to commute, we must have

\[ a_{ii}b_{ij} = a_{jj}b_{ji} \]

for all \( i, j \). This is true for our matrices except in the case where \( a_{ii} = -a_{jj} \). Thus, we can force commutativity if we assume that \( A \) has only positive (or only negative) eigenvalues.

5.2 \( R_A^3 = \pm R_{ABA} \) and \( R_B^3 = \pm R_{BAB} \) for Antisymmetric \( A, B \)

We wish to extend the above result to antisymmetric \( A, B \). This is nontrivial, since antisymmetric matrices cannot be diagonalized. They can, however, be block-diagonalized with 2-by-2 blocks down the diagonal with zeros elsewhere. Each of these blocks must be a scalar multiple of the matrix

\[ M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]
We’ll now formalize a process by which we use \( M \) to “diagonalize” an antisymmetric operator. It should be noted that a nontrivial antisymmetric operator cannot be diagonalized, since the diagonal entries of an antisymmetric matrix must be zero. In order to proceed, we will show that a block partitioning on an antisymmetric matrix preserves the antisymmetric properties of the matrix.

**Lemma 23.** Suppose \( B \) is a \( 2n \times 2n \) antisymmetric matrix over \( V \). Suppose we express \( B = [b_{ij}] \) as a block matrix, so that

\[
B = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1,n} \\
B_{21} & B_{22} & & B_{2,n} \\
& & \ddots & \\
B_{n,1} & B_{n,2} & \cdots & B_{n,n}
\end{bmatrix}
\]

where

\[
B_{ij} = \begin{bmatrix}
 b_{2i-1,2j-1} & b_{2i-1,2j} \\
 b_{2i,2j-1} & b_{2i,2j}
\end{bmatrix}
\]

Then, for all \( i, j \), we have

\[
B_{ij} = -(B_{ji})^T
\]

**Proof.** Let \( 1 \leq i, j \leq n \). Then,

\[
\begin{align*}
b_{2i-1,2j-1} &= -b_{2j-1,2i-1} \\
b_{2i,2j-1} &= -b_{2j-1,2i} \\
b_{2i-1,2j} &= -b_{2j,2i-1} \\
b_{2i,2j} &= -b_{2j,2i}
\end{align*}
\]

Thus, we have

\[
B_{ij} = \begin{bmatrix}
 b_{2i-1,2j-1} & b_{2i,2j-1} \\
 b_{2i,2j-1} & b_{2i,2j}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
 b_{2i-1,2j-1} & b_{2i,2j-1} \\
 b_{2i,2j-1} & b_{2i,2j}
\end{bmatrix}^T
\]

\[
= -\begin{bmatrix}
 b_{2j-1,2i-1} & b_{2j,2i-1} \\
 b_{2j,2i-1} & b_{2j,2i}
\end{bmatrix}^T
\]

\[
= -B_{ji}^T
\]

This directly shows that an antisymmetric operator on an even-dimensional \( V \) retains antisymmetry under block-partitioning. If \( n = \dim V \) is odd instead, then we can choose to partition the last row and last column of \( B \) with 1-by-1 matrices instead. Computation of matrix products is not hindered by this partitioning. We apply Lemma 23 to the newly partitioned block matrix without difficulty.

We now extend the result from Theorem 22 to antisymmetric operators.
Theorem 24. Let \( A = [a_{ij}] \), \( B = [b_{ij}] \) be \( n \times n \) antisymmetric matrices on a real, finite-dimensional vector space \( V \). Let \( \text{Rk}A \geq \text{RkB} \geq 4 \). Suppose \( A, B \) satisfy the following relationship:

\[
A^3B = \alpha AB^3
\]

for some \( \alpha \in \{1, -1\} \). Choose a basis for \( V \) so that \( A \) is block-diagonalized:

\[
A = \begin{bmatrix}
A_{11} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & A_{h,h} \\
0 & \ldots & 0 & 0
\end{bmatrix}
\]

where \( 2h = \text{Rk}A \) and

\[
A_{ii} = \begin{bmatrix}
0 & \lambda_i \\
-\lambda_i & 0
\end{bmatrix} = \lambda_i M
\]

where

\[
M = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

We partition \( B \) as a block matrix, so that \( B = [B_{ij}] \), where

\[
B_{ij} = \begin{bmatrix}
b_{2i-1,2j-1} & b_{2i-1,2j} \\
b_{2i,2j-1} & b_{2i,2j}
\end{bmatrix}
\]

and similarly \( B^3 = [\tilde{B}_{ij}] \). Then, for distinct indices \( i, j \), we have

\[
\lambda_i = \pm \lambda_j, \text{ or } B_{ij} = B_{ji} = [0]
\]

Proof. We partition \( B \) as a block matrix, so that \( B = [B_{ij}] \), where

\[
B_{ij} = \begin{bmatrix}
b_{2i-1,2j-1} & b_{2i-1,2j} \\
b_{2i,2j-1} & b_{2i,2j}
\end{bmatrix}
\]

and similarly \( B^3 = [\tilde{B}_{ij}] \). Note that since \( \text{RkB} \leq \text{Rk}A \), \( b_{ij} = 0 \) for \( i \geq 2h \) or \( j \geq 2h \).

Thus, \( B_{ij} = [0] \) for \( i \geq h \) or \( j \geq h \).

By hypothesis, we have

\[
A^3B = \alpha AB^3
\]

for some \( \alpha \in \{1, -1\} \). So,
\[ A^3 B = \begin{bmatrix}
  A_{11}^3 B_{11} & \ldots & A_{11}^3 B_{1h} & 0 & \ldots \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  A_{h,h}^3 B_{h,1} & \ldots & A_{h,h}^3 B_{h,h} & 0 & \ldots \\
  0 & \ldots & 0 & \ddots & \vdots \\
  \vdots & \ddots & \vdots & \ddots & \ddots \\
\end{bmatrix} = [A_{ii}^3 B_{ij}] \\
= \alpha AB^3 = \alpha \begin{bmatrix}
  A_{11}B_{11} & \ldots & A_{11}B_{1h} & 0 & \ldots \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  A_{h,h}B_{h,1} & \ldots & A_{h,h}B_{h,h} & 0 & \ldots \\
  0 & \ldots & 0 & \ddots & \vdots \\
  \vdots & \ddots & \vdots & \ddots & \ddots \\
\end{bmatrix} = \alpha [A_{ii}B_{ij}].
\]

So, for distinct indices \( i, j \) we have
\[
A_{ii}^3 B_{ij} = \alpha A_{ii} \tilde{B}_{ii} \quad \text{and} \quad A_{jj}^3 B_{ji} = \alpha A_{jj} \tilde{B}_{ji}.
\]

\[ \implies \lambda_i^3 M^3 B_{ij} = \alpha \lambda_i M \tilde{B}_{ij} \quad \text{and} \quad \lambda_j^3 M^3 B_{ji} = \alpha \lambda_j M \tilde{B}_{ji}, \]

since \( M^4 = I \) and \( \lambda_k \neq 0 \) for \( k \leq h \). But by Lemma 23, we have
\[
\tilde{B}_{ij} = -\tilde{B}_{ji}^T,
\]

and so
\[
\lambda_i^2 M^2 B_{ij} = \alpha \tilde{B}_{ij} = -\alpha \tilde{B}_{ji}^T = -\lambda_j^2 (M^2 B_{ji})^T
\]

\[ \implies -\lambda_i^2 B_{ij} = (-\lambda_j^2)(-B_{ji})^T = \lambda_j^2 (M^2 B_{ji})^T \quad \text{(since } M^2 = -I) \]

\[ \implies -\lambda_i^2 B_{ij} = -\lambda_j^2 B_{ij} \]

by Lemma 23. So, we have
\[
B_{ij} = 0 \text{ or } \lambda_i = \pm \lambda_j.
\]

6 Chain Complexes as Directed Graphs

We wish to determine the number of distinct chain complexes on a given number of operators. We find that associating a graph with each chain complex gives us a better language to count the complexes. For a compound chain complex on a vector space \( V \), we wish to associate each instance of \( V \) with a distinct vertex and associate the operators on \( V \) to edges between vertices that preserve the component complexes. To allow for a unique association between a compound chain complex and a directed graph, we need notions of a source and a sink.
Definition 25. Let $G$ be a finite directed graph. We call a vertex $v \in G$ a **source** if $v$ receives no edges in $G$. We call $v$ a **sink** if $v$ sends no edges in $G$.

Since it is possible that in a compound chain complex, there are many different “sources” (in the way that different component complexes may begin with different operators) and different “sinks” (in the way that different component complexes may end with different operators), we unify “source” spaces and unify “sink” spaces so that no graph associated to a chain complex has more than one source vertex or sink vertex.

In general, for a chain complex, there are potentially many graphs that could represent the complex, and there does not seem to be a natural way to identify these graphs with one another. However, for complexes with 4 or fewer edges, the representations are determined entirely by the unification of sinks and sources, as above.

Additionally, we wish only to consider graphs which yield a valid complex. Necessary criteria for determining whether a graph $G$ yields a valid complex are as follows:

1. $G$ is connected
2. $G$ has at most one source and at most one sink, for the reasons listed above.
3. If $v, u$ are vertices in $G$, then there is at most one edge in $G$ from $u$ to $v$.
4. The chain complex that is formed from $G$ has a proper linear dependence equation for the canonical algebraic curvature tensors. Equivalently, the graph does not force any operator in the complex to be identically 0.

The above list does not exhaust the possible criteria for determining all graphs from which a chain complex could be drawn. However, it gives us a good starting point for the discussion of graphs as chain complexes.

6.1 Graphs with 3 or Fewer Edges

According to the above criteria, there are no valid graphs on 2 edges. Moreover, there is only one valid graph on 3 edges:

\[
\begin{array}{ccc}
\circ & \text{A} & \circ \\
\text{B} & \circ & \text{C} \\
\circ & \circ & \circ
\end{array}
\]

For 3-edge graphs, the associated linear equation has the form

\[ R_A + \alpha_1 R_B + \alpha_2 R_C = 0 \]

for $\alpha_1, \alpha_2 \in \{1, -1\}$. Under precomposition by operators, every other graph on 3 edges requires at least one operator to be 0.
6.2 4-Edge Graphs

The number of valid graphs on 4 edges is substantially larger than that on 3. We document the valid graphs here without elaborating on the properties of any individual graph. For 4-edge graphs, the associated linear dependence equation has the form

$$R_A + \alpha_1 R_B + \alpha_2 R_C + \alpha_3 R_D = 0$$

for $\alpha_1, \alpha_2, \alpha_3 \in \{1, -1\}$

Some 4-edge graphs correspond to complexes already studied by McMahon in [5]. We catalogue them here without references to results.

To see results on the following graph, see Chapter 3:

All other graphs belong to one of two groups: mostly linear graphs and mostly cyclical graphs. We develop no results on these graphs, but list them here for reference.
6.2.1 Mostly Linear Graphs

6.2.2 Mostly Cyclical Graphs

7 Questions and Projects for Further Study

1. For \( n \geq 5 \), classify directed graphs of \( n \) edges/operators that can be associated with some chain complex. Impose a linear dependence on the operators in the chain complex, and don’t consider graphs which force any operator to be 0 (these are not useful for our purposes). For valid graphs on \( n \geq 4 \) edges/operators, form the hierarchy of graphs from least restrictive to most restrictive. For example, on 4 edges/operators, there are 14 valid graphs, some of which are stricter than others. Which ones are the least strict? Which are the most strict? If we start with a graph which is not very strict and impose more containment relationships on the images and kernels of the operators, what other graphs can we derive?
What are the possible restriction paths we could take from a least-strict graph to a most-strict graph? Answer these questions for $4 \leq n \leq 10$ edges/operators.

2. For a set of CACT’s which is known to be linearly dependent, consider what other conditions are necessary for us to conclude that a chain complex structure must exist on the underlying operators. This is intended to fit McMahon’s and my work with chain complexes into the greater discussion about linearly dependent sets of ACT’s, since satisfying a chain complex structure seems to be a strong condition. Other problems along this vein are: For an ACT $R$, how does chain complex analysis interact with our knowledge about $\nu(R)$, $\eta(R)$, and $\mu(R)$ (see [5] for definitions and discussions about these values)? For a given vector space $V$, can we use our knowledge about $\mathcal{A}(V)$ to restrict the kinds of possible chain complexes on $V$? Some of my work is very similar to the work that other REU students have done, and I think there are ways to combine my methods and theirs to acquire a fuller toolbox for analyzing ACT’s.

3. Find classes of solutions for common relationships found in chain complexes, such as

$$R_{A^3} + \alpha R_{ABA} + \delta R_{ACA} = 0$$

for $A, B, C$ symmetric or anti-symmetric and for $\alpha, \delta \in \{1, -1\}$. Examine these equations as matrix polynomials and also as systems of equations, probably with very many unknowns. This seems like a very tedious problem that may involve heavy use of CAS, but getting information about the the properties or entries of these matrices could help us characterize the solutions to a given chain complex. Partitioning this problem into cases based on rank assumptions seems to be the most logical way to progress.

4. Reexamine the work Elise and I have done with the new assumption that all sequences are exact. This has the effect of making every containment assumption an equality assumption instead. How does this change the number of valid graphs on 4 or more operators/edges? Are the solutions to matrix equations more readily derived? Are the linear dependence equations easier to work with? Many cases should become trivial, and nontrivial examples seem like they should simplify immensely. With exact sequences, is it possible to make concrete statements about sets of higher numbers of tensors?

5. Use homology theory to dissect the kernels of operators in chain complexes, and determine if there is a clear connection between homologies and operators. For sequences that are necessarily not exact, there is potential variance in the size of the operator’s image. Just as I was able to draw many conclusions by examining the kernels of the operators, it might be possible to derive new conclusions by studying the homologies of the operators. Additionally, it should be noted that my assumption that the intersection of the kernels of the operators is trivial is equivalent to assuming that the intersection of the homologies of the operators is trivial in the chain complex.
6. The convenient (and restrictive) property of a chain complex is that if A immediately precedes B in complex, BA = 0. This is certainly sufficient to show that $R_{ABA} = 0 = R_{BAB}$, but it is not necessary. A less restrictive (and so more plausible) condition is to require that $RkABA \leq 1 \geq RkBAB$, which is necessary to show $R_{ABA} = 0 = R_{BAB}$. For a given A, define the flattening of A, denoted $F(A)$, to be the set of matrices such that $RkABA \leq 1 \geq RkBAB$. For a given A, what kind of set is $F(A)$? Is it ever a group? An abelian group? Which elements $B \in F(A)$ satisfy $ImA \subset KerB$? Furthermore, when we use this new condition to form pseudo-chain-complexes, how does this change the properties of the operators? Are the implications similar or different from the results on normal chain complexes? How does this new assumption affect the matrix equations and graphical classifications?

7. Consider more abstract graph-theoretic properties of the directed graphs that are derived from chain complexes. If the graphs of two chain complexes are duals of each other, is there any connection to the complexes themselves? Is there a canonical/logical flow on the edges of a directed graph which corresponds somehow to the operators on the underlying chain complex? Also, it seems that the number of valid graphs which contain no directed cycles is fairly low for every number of operators. Why is this? Is there any way to “replace” a directed cycle in a graph with a non-cycle construct that maintains the linear dependence of the operators? Consider the polarization formula and matrix splitting, as well as Elise’s identity and other identities for tensors and matrices.

8. For symmetric matrix A and anti-symmetric matrices C, D find general forms or classifications for the products $C^3D$, $CD^3$, $C^3A$, $CA^3$, $A^3C$, $AC^3$, $ABAB$, $ACAC$, $CACA$, $CDCD$. These operators appear frequently in working with precomposed ACT’s, but their properties are tedious to determine in general.

9. Reexamine my work by restating my results in terms of the images and ranks of operators rather than kernels and nullities. How does this change the underlying assumptions about our operators? Does this make any of my results more intuitive? Less intuitive?

10. Given operators A, B on V, find relations for $ImAB$, $RkAB$, $KerAB$, $NlAB$ in terms of $ImA$, $RkA$, $KerA$, $NlA$, $ImB$, $RkB$, $KerB$, and $NlB$.

11. Determine whether the work Elise and I have done holds for cases in which some the ranks of the operators are exactly 2. I often avoided studying cases in which the ranks of the operators were exactly 2, as there were few previous results that applied to them. Do my results still hold? Do Elise’s?

12. Let M be defined as follows.

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
This matrix is notable because it allows us to express an antisymmetric block-diagonal matrix as a diagonal block matrix. Additionally, M has several properties that are very similar to the imaginary number $i$: $M^2 = -I$ for example, and $M = -M^3$. I have a hunch that by studying $M$ alongside antisymmetric matrices, there may be a way to express antisymmetric matrices in a more convenient way. By my proof of Theorem 24, we know that there exists a function $\phi$ which associates a block-diagonal antisymmetric matrix with a unique diagonal matrix. Ideally, we would be able to expand $\text{Dom}(\phi)$ to all antisymmetric matrices and $\text{Im}(\phi)$ to all symmetric matrices so that $\phi$ is a bijection. If such a $\phi$ is found, many previous results (including mine) could be simplified to just the symmetric case, and we would have a new tool with which to study canonical algebraic curvature tensors.

13. For a given compound chain complex $\mathcal{D}$, let $v(\mathcal{D})$ be the smallest value of $\dim V$ that will allow for the sequences to be nowhere exact given the constraints of Theorem 20. Let $\Upsilon(n) = \max\{v(\mathcal{D}) : \mathcal{D} \text{ is a compound chain complex on } n \text{ operators}\}$. Determine $\Upsilon(4)$, and begin estimations on $\Upsilon(5)$.

8 Acknowledgments

I would like to thank Drs. Corey Dunn and Rolland Trapp for their insight, guidance, and dedication to their students. Also a big thank you to California State University, San Bernardino for the use of its facilities. This research was supported by the National Science Foundation grant DMS-1461286.

References


