General Curvature Homogeneity Theories
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Abstract
This paper studies past notions of curvature homogeneity while also discussing $G$-modeled pseudo-Riemannian manifolds. It is shown that for any $m \in \mathbb{N} \cap [3, \infty)$, there exists a manifold $M$ of dimension $m$, a model space $\mathcal{M}$, and a one dimensional Lie group such that $M$ is $G$-modeled up to order 0 with respect to $\mathcal{M}$. In addition, the manifold fails to be curvature homogeneous as well as homothety curvature homogeneous.

1 Introduction
Past studies of curvature homogeneity, for example in [1], have involved the comparison of pseudo-Riemannian manifolds and model spaces. This paper dives into a deeper theory of curvature homogeneity, which was first studied in [3]. We wish to examine a new branch of manifolds known as $G$-modeled manifolds. This theory involves the ingredients of a pseudo-Riemannian manifold, a model space, and a Lie group $G$. One of the first examples of a $G$-modeled manifold (see [3]) appears in the form of a three dimensional manifold being $G$-modeled with respect to a model, and a one dimensional Lie group $G$. In this paper, we extend this particular result by increasing the dimension of the manifold to any arbitrary finite dimension $m$, while leaving the dimension of the Lie group the same.

1.1 Manifolds and Model Spaces
For the remainder of the paper, when a vector space is mentioned, we assume that it is real and finite dimensional.

Definition 1. Let $V$ be a vector space. A function $\phi : V \times V \to \mathbb{R}$ is called an inner product on $V$ if

1. $\phi$ is bilinear,

2. $\phi(v, w) = \phi(w, v)$ for any $v, w \in V$,

3. $\phi(v, v) > 0$ if $v \neq 0$ (positive definite).

Furthermore, if $M$ is a manifold, then a metric $g$ on $M$ is a choice of inner product on each tangent space. We denote the tangent space at a point $P \in M$ as $T_PM$. We say that the tuple $(M, g)$ is a pseudo-Riemannian manifold if $g$ admits an inner product on each $T_PM$ that satisfies the following property:
4. For any \( v \in V \setminus \{0\} \), there exists a \( w \in V \) such that \( \phi(v, w) = 0 \).

Given a pseudo-Riemannian manifold \((M, g)\) as discussed above, if \( \nabla \) is Levi-Civita connection on \( M \), we define the Riemannian curvature tensor \( R \) on the vector fields \( X, Y, Z, W \) as

\[
R(X, Y, Z, W) := g \left( \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \right).
\]

Furthermore, we define \( \nabla^i R \) to be the \( i \)th covariant derivative of \( R \). If \( P \) is a point on \( M \), then we denote \( T_P M \) to be the tangent space at \( P \), and \( g_P, \nabla^i R_P \) to be the metric \( g \) and \( i \)th covariant derivative at \( P \), respectively.

Definition 2. A map \( R_0 : V^4 \to \mathbb{R} \) is called an algebraic curvature tensor on \( V \) if

1. \( R_0 \) is multilinear,
2. \( R_0(x, y, z, w) = -R_0(y, x, z, w) \),
3. \( R_0(x, y, z, w) = R_0(z, w, x, y) \), and
4. \( R_0(x, y, z, w) + R_0(z, x, y, w) + R_0(y, z, x, w) = 0 \),

for all \( x, y, z, w \in V \). The set of algebraic curvature tensors on \( V \) is denoted by \( \mathcal{A}(V) \). The tuple \((V, \phi, R_0)\) is called a model space. For simplicity, we denote the set of all model spaces over \( V \) as \( \mathcal{M}(V) \).

Remark 1. We note that in the above definition, \( R_0 \) was an element of \( \otimes^4 V^* \) that satisfies the same algebraic properties as the Riemannian curvature tensor. We may further extend the definition above by saying that a model space is a tuple \((V, \phi, R_0, R_1, \ldots, R_k)\) where \( V \) is a vector space, \( \phi \) is an inner product on \( V \), and each \( R_i \), for \( i = 0, 1, \ldots, k \), is an element of \( \otimes^{4+i} V^* \) that satisfies the same algebraic properties as \( \nabla^i R \).

Given two model spaces \( \mathcal{V} = (V, \phi_1, R_0, \ldots, R_k) \) and \( \mathcal{W} = (W, \phi_2, S_0, \ldots, S_k) \), we say that \( \mathcal{V} \) is isomorphic to \( \mathcal{W} \) and write \( \mathcal{V} \cong \mathcal{W} \), if there exists an invertible linear map \( A : V \to W \) such that \( \mathcal{V} = A^* \mathcal{W} \), where \( A^* \) represents precomposition by \( A \), and

\[
A^* W := (W, A^* \phi_2, A^* S_0, \ldots, A^* S_k).
\]

Remark 2. It is important to note that in some cases, for a model space \( \mathcal{M} \), we sometimes define \( A^* \mathcal{M} \) exactly as above, but do not precompose \( A \) with the inner product.

1.2 Curvature Homogeneity Theories

The next definitions discuss the types of curvature homogeneity we wish to study. If a model is of the form \( \mathcal{M} = (V, \phi, R_0, \ldots, R_k) \), then we say that \( \mathcal{M} \) is a \( k \)-model and in some cases use the notation \( \mathcal{M}^k \) in its place. Also, if \((M, g)\) is a pseudo-Riemannian manifold and \( P \in M \), then the tuple

\[
\mathcal{M}_P^k := (T_PM, g_P, R_P, \ldots, \nabla^k R_P)
\]

is a \( k \)-model space.
Definition 3. Let \((M, g)\) be a pseudo-Riemannian manifold and let \(\mathcal{M}^k = (V, \phi, R_0, \ldots, R_k)\) be a model space of order \(k\). Then, we say that \((M, g)\) is:

- **Curvature homogeneous up to order \(k\) (\(CH_k\)) with model \(\mathcal{M}^k\), if \(\mathcal{M}^k_P \cong \mathcal{M}^k\) for every \(P \in M\).

- **Homothety curvature homogeneous up to order \(k\) (\(HCH_k\)) with model \(\mathcal{M}^k\), if there exists a nonzero smooth function \(\lambda : M \rightarrow \mathbb{R}\) such that for every \(P \in M\),
  \[
  \mathcal{M}^k_P \cong (V, \phi, \lambda R_0, \lambda^3 R_1, \ldots, \lambda^{k+2} R_k).
  \]

Remark 3. It is clear to see that \(CH_k \Rightarrow HCH_k\) by simply setting \(\lambda = 1\).

While the above types of curvature homogeneity have been studied in the past, we can generalize these notions by also considering a Lie group that acts on the set of model spaces over \(V\). Adding this aspect of a group action has only recently been studied in [3]. This idea is clearly laid out in the following definition.

Definition 4. Let \((M, g)\) be a pseudo-Riemannian manifold and \(\mathcal{M} = (V, \phi, R_0, \ldots, R_k)\) be a model space. Suppose that \(G \leq \text{Gl}(V)\) is a Lie group and \(A \mapsto A \cdot \mathcal{M}\) is an action of \(G\) on the set of model spaces over \(V\). Then, \((M, g)\) is \(G\)-modeled up to order \(k\) provided the following hold:

1. For every \(P \in M\) there exists an \(A \in G\) such that \(\mathcal{M}^k_P \cong A \cdot \mathcal{M}\).

2. For every \(A \in G\) there exists a \(P \in M\) such that \(\mathcal{M}^k_P \cong A \cdot \mathcal{M}\).

1.3 Invariants of Curvature Homogeneous Manifolds

As briefly discussed before, we are interested in manifolds that live in the category of Definition 4 but not that of Definition 3. The following result is a useful curvature homogeneous invariant, which was first used in [2] and [3].

Proposition 1. Suppose \((M, g)\) is an \(HCH_0\) manifold with model \(\mathcal{M}\). Define

\[
\tau := \sum_{i,j,k,l} g^{il}g^{jk}R_{ijkl} \quad \text{and} \quad ||R||^2 := \sum_{i_1,j_1,,i_4,j_4} g^{i_1j_1}g^{i_2j_2}g^{i_3j_3}g^{i_4j_4}R_{i_1i_2i_3i_4}R_{j_1j_2j_3j_4},
\]

then we have that

\[
\frac{\tau(P_1)^2}{||R(P_1)||^2} = \frac{\tau(P_2)^2}{||R(P_2)||^2}
\]

for any points distinct points \(P_1, P_2 \in M\).

We use the contrapositive of this statement, in practice.

Corollary 1. Let \((M, g)\) be a manifold. As defined above, if \(\frac{\tau^2}{||R||^2}\) is non-constant on \(M\), then \((M, g)\) is not \(HCH_0\), and hence not \(CH_0\) either.
2 \quad \textit{G}-modeled Manifolds of 4 Dimensions

In [3], it was shown that there exists a 3-dimensional manifold that is \( G \)-modeled up to order 0, with respect a model space \( \mathcal{M} \) and a Lie group \( G \cong \mathbb{R}^+ \) acting on \( \mathcal{M} \). This manifold was also not \( HCH_0 \) nor \( CH_0 \). While it is our overall goal to construct an arbitrary finite dimensional manifold satisfying these conditions, we begin by showing that there exists such a manifold of 4 dimensions.

Let \( M := \{(x_1, x_2, x_3, x_4) : x_1 > 0\} \), where \((x_1, x_2, x_3, x_4)\) are the standard coordinates of \( \mathbb{R}^4 \) and define a metric on \( M \) via
\[
g(\partial x_1, \partial x_1) = 1, \quad g(\partial x_2, \partial x_3) = g(\partial x_2, \partial x_4) = e^{2f(x_1)}, \quad \text{and} \quad g(\partial x_3, \partial x_3) = h(x_1)
\]
where we define \( f(x_1) := -x_1 + \ln(e^{x_1} - 1) \) and \( h(x_1) := \frac{1}{4}e^{2x_1} \). Let \( V = \text{span}\{X_1, X_2, X_3, X_4\} \).

We construct a model space on \( V \), and define it to be \( \mathcal{M} := (V, \phi, R_0) \) where the following are the nonzero entries of \( R_0 \) and \( \phi \), up to the standard symmetries:
\[
\phi(X_1, X_1) = \phi(X_2, X_3) = \phi(X_2, X_4) = 1,
\]
\[
R_0(X_1, X_3, X_3, X_1) = R_0(X_3, X_4, X_2, X_3) = -1,
\]
and
\[
R_0(X_1, X_2, X_3, X_1) = R_0(X_2, X_3, X_3, X_2) = R_0(X_1, X_2, X_4, X_1) = R_0(X_2, X_4, X_4, X_2) = 1.
\]
Let \( G \leq GL(V) \) be the Lie group
\[
G = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t > 0 \right\},
\]
which is isomorphic to \( \mathbb{R}^+ \). Let \( G \) act on the set of model spaces over \( V \) by
\[
G \times \mathcal{M}(V) \to \mathcal{M}(V)
\]
\[
(A, (V, \phi, R_0)) \mapsto (V, \phi, A^*R_0).
\]
The following Lemma is due to calculations done in Maple.

**Lemma 1.** Let \((M, g)\) be constructed as above. The following hold:

1. The nonzero covariant derivatives of the coordinate frames are
\[
\nabla_{\partial x_2} \partial x_3 = \nabla_{\partial x_3} \partial x_2 = \nabla_{\partial x_2} \partial x_4 = \nabla_{\partial x_4} \partial x_2 = -e^{2f(x_1)} f'(x_1) \partial x_1,
\]
\[
\nabla_{\partial x_3} \partial x_3 = -\frac{h'(x_1)}{2} \partial x_1, \quad \nabla_{\partial x_3} \partial x_2 = \nabla_{\partial x_2} \partial x_1 = f'(x_1) \partial x_2,
\]
\[
\nabla_{\partial x_3} \partial x_4 = \nabla_{\partial x_4} \partial x_3 = \frac{h'(x_1)}{2h(x_1)} \partial x_3 + \left( -\frac{h'(x_1)}{2h(x_1)} + f'(x_1) \right) \partial x_4,
\]
and
\[
\nabla_{\partial x_3} \partial x_4 = \nabla_{\partial x_4} \partial x_1 = f'(x_1) \partial x_4.
\]
2. The nonzero curvature entries (up to the usual symmetries) are

\[ R(\partial x_1, \partial x_3, \partial x_3, \partial x_1) = \frac{(h'(x_1))^2}{4h(x_1)} - \frac{h''(x_1)}{2}, \quad R(\partial x_3, \partial x_4, \partial x_2, \partial x_3) = -\frac{e^{2f(x_1)}(f'(x_1))(h'(x_1))}{2}, \]

\[ R(\partial x_2, \partial x_3, \partial x_3, \partial x_2) = R(\partial x_2, \partial x_4, \partial x_4, \partial x_2) = e^{4f(x_1)}(f'(x_1))^2, \]

and

\[ R(\partial x_1, \partial x_2, \partial x_3, \partial x_1) = R(\partial x_1, \partial x_2, \partial x_4, \partial x_1) = -e^{2f(x_1)}((f'(x_1))^2 + f''(x_1)). \]

**Theorem 1.** As defined above, \((M, g)\) is \(G\)-modeled up to order 0 with respect to the model space \(\mathcal{M}\) (and group action as above). Moreover, \((M, g)\) is neither \(CH_0\) nor \(CH_0\).

**Proof.** Using Lemma 1 part 2, we consider the frame \(\{X_1, X_2, X_3, X_4\}\) (note that this is an abuse of notation when considering \(V\) above, but the significance of this abuse will become clear) where

\[ X_1 = \partial x_1, \quad X_2 = \frac{\sqrt{\Delta}}{e^{2f(x_1)}} \partial x_2, \quad X_3 = \frac{1}{\sqrt{\Delta}} \partial x_3, \quad \text{and} \quad X_4 = \frac{1}{\sqrt{\Delta}} \partial x_4. \]

Here, we define \(\Delta := \frac{(h'(x_1))^2}{4h(x_1)} - \frac{h''(x_1)}{2}\). It is an easy verification that

\[ |\Delta| = -\Delta \quad \text{and} \quad \frac{h'(x_1)}{2} = |\Delta| \]

for any \(x_1 > 0\). Now,

\[ g(X_1, X_1) = g(X_2, X_3) = g(X_2, X_4) = 1 \]

and

\[ R(X_1, X_3, X_3, X_1) = \frac{\Delta}{|\Delta|} = -1, \quad R(X_1, X_2, X_3, X_1) = R(X_1, X_2, X_4, X_1) = -((f'(x_1))^2 + f''(x_1)), \]

\[ R(X_2, X_3, X_3, X_2) = R(X_2, X_4, X_4, X_2) = (f'(x_1))^2, \]

\[ \text{and} \quad R(X_3, X_2, X_4, X_3) = -\frac{f'(x_1)h'(x_1)}{2|\Delta|} = -f'(x_1). \]

It is trivial to see that \(f'(x_1) = -((f'(x_1))^2 + f''(x_1))\) and also that \(f'(\{x_1 : x_1 > 0\}) = \mathbb{R}^+\). Hence, for any \(t > 0\), we can find a point \(P_0 = (y_1, y_2, y_3, y_4) \in M\) such that \(f'(y_1) = t\), and when considering the frame \((X_1, X_2, X_3, X_4)\), we have

\[ R(X_1, X_3, X_3, X_1) = -1, \quad R(X_1, X_2, X_3, X_1) = R(X_1, X_2, X_4, X_1) = t, \]

\[ R(X_2, X_3, X_3, X_2) = R(X_2, X_4, X_4, X_2) = t^2, \quad R(X_3, X_2, X_4, X_3) = -t. \]

In other words, given any \(A \in G\), there exists a point \(P \in M\) such that \(R_P = A^* R_0\) and hence \(A^* M = M_P\). We conclude that \((M, g)\) is \(G\)-modeled up to order 0 with model \(\mathcal{M}\). To verify that this manifold is not \(CH_0\), we calculate the values in (1) (via Maple) as

\[ \tau = \frac{2(-e^{2x_1} + 2e^{x_1} - 2)}{(e^{x_1} - 1)^2} \]
and
\[ ||R||^2 = \frac{16(36e^{x_1} - 131e^{4x_1} - 32e^{6x_1} + 80e^{5x_1} - e^{8x_1} + 8e^{7x_1} - 94e^{2x_1})}{(e^{x_1} - 1)^2} \]
for any \( x_1 > 0 \). Now, one could verify that \( \frac{r^2}{||R||^2} \) is non-constant as \( x_1 \) varies. Thus, by Corollary 1, we find that our manifold is not \( HCH_0 \). By Remark 3, it is also not \( CH_0 \). □

3 \ G-Modeled Manifolds of Finite Dimension \( m \geq 3 \)

It is our goal to prove the following Theorem:

**Theorem 2.** Let \( m \in \mathbb{N} \cap [3, \infty) \). Then, there exists:

- a pseudo-Riemannian manifold \((M, g)\),
- a 0-model \( \mathcal{M} \), and
- a Lie group \( G \cong \mathbb{R}^+ \) and an action of \( G \) on \( \mathcal{M}(V) \)

such that \( M \) is \( G \)-modeled up to order 0 with respect to the model space \( \mathcal{M} \), and \( \dim(M) = m \). Furthermore, \((M, g)\) is neither \( CH_0 \) nor \( HCH_0 \).

For the remainder of this section, suppose that \( n \) is an arbitrary element of \( \mathbb{N} \).

3.1 Construction for \( m = 2n + 1 \)

**Lemma 2.** Let \( M = \mathbb{R}^{2n+1} \) with coordinates \((x_1, x_2, \ldots, x_{2n+1})\). Suppose \( f \) is a function of only \( x_1 \). Let \( g \) be a metric on \( M \) with nonzero entries given by
\[ g(\partial x_1, \partial x_1) = 1 \]
and
\[ g(\partial x_2, \partial x_3) = g(\partial x_4, \partial x_5) = \cdots = g(\partial x_{2n}, \partial x_{2n+1}) = e^{2f(x_1)}. \]

The following hold:

1. The nonzero covariant derivatives of the coordinate frames are
\[ \nabla_{\partial x_1} \partial x_k = \nabla_{\partial x_k} \partial x_1 = f'(x_1) \partial x_k \text{ and } \nabla_{\partial x_i} \partial x_j = \nabla_{\partial x_j} \partial x_j = -f'(x_1)e^{2f(x_1)} \partial x_1 \]
where \( k = 2, 3, \ldots, 2n + 1 \), and \( \{i, j\} \in U := \{\{i, j\} : g_{x_ix_i} = g_{x_jx_j} = e^{2f(x_1)}\} \).

2. The nonzero curvature entries up to symmetry are
\[ R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = -e^{2f(x_1)}((f'(x_1))^2 + f''(x_1)), \quad R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) = e^{4f(x_1)}(f'(x_1))^2 \]
and
\[ R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = -e^{4f(x_1)}(f'(x_1))^2, \]
where \( \{i, j\} \) and \( \{a, b\} \) are unique sets contained in \( U \).
Proof. 1. By construction, the only nonzero Christoffel symbols of the second kind (up to symmetries) are

\[ \Gamma_{1kl} = f'(x_1)e^{2f(x_1)} \text{ and } \Gamma_{kl1} = -f'(x_1)e^{2f(x_1)}, \]

where \( \{k, l\} \in U \). Now, since

\[ \Gamma_{1kl} = g \left( \sum_{m=1}^{2n+1} \Gamma_{1m} \partial x_m, \partial x_l \right) = \Gamma_{1k}^l g(\partial x_k, \partial x_l), \]

we have that

\[ \Gamma_{1k}^m = \begin{cases} f'(x_1) & \text{if } m = k \\ 0 & \text{otherwise} \end{cases}. \]

This implies that \( \nabla_{\partial x_k} \partial x_k = f'(x_1)\partial x_k \) and similar derivations show that \( \nabla_{\partial x_k} \partial x_1 \) obtains the same value. Also, we have that

\[ \Gamma_{kl1} = g \left( \sum_{m=1}^{2n+1} \Gamma_{klm} \partial x_m, \partial x_1 \right) = \Gamma_{kl1}^1 g(\partial x_1, \partial x_1), \]

and hence

\[ \Gamma_{kl1}^1 = \begin{cases} -f'(x_1)e^{2f(x_1)} & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}. \]

Thus, \( \nabla_{\partial x_k} \partial x_1 = -f'(x_1)e^{2f(x_1)}\partial x_1 \) and by symmetry, we conclude that \( \nabla_{\partial x_k} \partial x_k \) obtains the same value, proving the second assertion of (1). It is also clear that any other covariant derivative entries vanish.

2. Let \( \{i, j\}, \{a, b\} \in U \). By part 1, we have

\[
R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = g \left( \nabla_{\partial x_1} \nabla_{\partial x_i} \partial x_j - \nabla_{\partial x_i} \nabla_{\partial x_1} \partial x_j, \partial x_1 \right)
= g \left( -2(f'(x_1))^2 e^{2f(x_1)} - f''(x_1)e^{2f(x_1)} - (f'(x_1))^2 e^{2f(x_1)}, \partial x_1 \right)
= -((f(x_1))^2 + f''(x_1)),
\]

\[
R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) = g \left( \nabla_{\partial x_i} \nabla_{\partial x_j} \partial x_j - \nabla_{\partial x_j} \nabla_{\partial x_i} \partial x_j, \partial x_i \right)
= g((f'(x_1))^2 e^{2f(x_1)} \partial x_j, \partial x_i)
= -e^{4f(x_1)}(f'(x_1))^2,
\]

and

\[
R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = g \left( \nabla_{\partial x_i} \nabla_{\partial x_a} \partial x_b - \nabla_{\partial x_a} \nabla_{\partial x_i} \partial x_b, \partial x_j \right)
= g \left( -(f'(x_1))^2 e^{2f(x_1)} \partial x_i, \partial x_j \right)
= -e^{4f(x_1)}(f'(x_1))^2,
\]

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as needed. It is left to show that the remaining curvature entries vanish. Let \( p, q, r, s \in \{2, 3, \ldots, 2n + 1\} \). First assume that \( \{q, r\} \not\in U \). Then,

\[
R(\partial x_p, \partial x_q, \partial x_r, \partial x_s) = g \left( -\nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r, \partial x_s \right)
\]

\[
= \left\{ \begin{array}{ll}
g \left( (f'(x_1))^2 e^{2f(x_1)} \partial x_p, \partial x_s \right) & \text{if } \{p, r\} \in U \\
0 & \text{otherwise} \end{array} \right.
\]

However, we note that if both \( \{p, r\}, \{q, s\} \in U \), then this is a symmetry of a nonzero curvature entry. Thus, if either \( \{p, r\} \) or \( \{q, s\} \) are not in \( U \), then this curvature entry vanishes, as required. Now suppose that \( \{q, r\} \in U \). To distinguish from an existing nonzero curvature entry, it must be the case that \( \{p, s\} \not\in U \). Now,

\[
R(\partial x_p, \partial x_q, \partial x_r, \partial x_s) = g \left( -(f'(x_1))^2 e^{2f(x_1)} \partial x_p - \nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r, \partial x_s \right) = 0,
\]

where the last inequality holds since \( g(\partial x_p, \partial x_s) = 0 \) and \( \nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r = 0 \). It remains to verify that any curvature entry with only one input of \( \partial x_1 \) is zero. Let \( p, q, r \in \{2, 3, \ldots, 2n + 1\} \). We have that

\[
R(\partial x_1, \partial x_p, \partial x_q, \partial x_r) = \left\{ \begin{array}{ll} 
g \left( -e^{2f(x_1)} ((f'(x_1))^2 + f''(x_1)) \partial x_p, \partial x_s \right) & \text{if } \{p, r\} \in U \\
g \left( -e^{2f(x_1)} ((f'(x_1))^2 + f''(x_1)) \partial x_q, \partial x_s \right) & \text{otherwise} \end{array} \right.
\]

but since \( g(\partial x_1, \partial x_s) = 0 \), this value is zero in either case. Due to symmetry, we have that any curvature entry with only one input of \( \partial x_1 \) is zero.

\[\square\]

The goal now is to utilize the metric construction of Lemma 2 on a particular manifold, and show that it is \( G \)-modeled up to order 0 for some Lie group isomorphic to \( \mathbb{R}^+ \). In addition, we want our construction to omit a manifold which is neither \( CH_0 \) nor \( HCH_0 \). We now show given any \( 2n + 1 \), there exists a manifold \( M \) satisfying the above properties with \( \dim(M) = 2n + 1 \).

Let \( (x_1, x_2, \ldots, x_{2n+1}) \) be the standard coordinates of \( \mathbb{R}^{2n+1} \) and define

\[
M := \{(x_1, x_2, \ldots, x_{2n+1}) | x_1 > 0\}
\]

to be our manifold. We also let our metric \( g \) on \( M \) to be defined as in Lemma 2, while using the same notation for \( U \), and also setting \( f(x_1) := -x_1 + \ln(e^{x_1} - 1) \). With this mind, we let our model space be given by \( \mathcal{M} := (V, \phi, R_0) \) where \( V = \text{span}\{X_1, X_2, \ldots, X_{2n+1}\} \), and the nonzero inner product entries (up to the standard symmetries) are given by

\[
\phi(X_1, X_1) = \phi(X_2, X_3) = \cdots = \phi(X_{2n}, X_{2n+1}) = 1.
\]

If we denote \( \bar{U} := \{\{i, j\} : \phi(X_i, X_j) = \phi(X_j, X_i) = 1 \text{ and } i \neq j\} \), then we define the nonzero algebraic curvature entries as

\[
R_0(X_1, X_i, X_j, X_1) = R_0(X_i, X_j, X_j, X_i) = 1, \quad R_0(X_i, X_a, X_b, X_j) = -1.
\]
where \(\{i, j\}\) and \(\{a, b\}\) are distinct sets in \(\bar{U}\). We also let \(G \leq Gl(V)\) be the Lie group defined to be the set of \((2n + 1) \times (2n + 1)\) matrices \(A = [a_{ij}]\) for which

\[
a_{ij} = \begin{cases} 
1 & \text{if } i = j \equiv 1 \pmod{2}, \\
t & \text{if } i = j \equiv 0 \pmod{2}, \\
0 & \text{otherwise}
\end{cases}
\]

where \(t \in \mathbb{R}^+\), so that \(G \cong \mathbb{R}^+\). Furthermore, we let \(G\) act on the set of model spaces over \(V\) (call this set \(\mathcal{M}(V)\)) via

\[
G \times \mathcal{M}(V) \to \mathcal{M}(V)
\]

\[
(A, (V, \phi, R_0)) \mapsto (V, \phi, A^* R_0),
\]

where \(A^*\) represents precomposition by \(A\).

\textbf{Theorem 3.} \textit{As defined above, the manifold \((M, g)\) is \(G\)-modeled up to order 0 with respect to the model space \(\mathcal{M}\) and the given group action on \(\mathcal{M}(V)\). In addition, \((M, g)\) is not HCH\(_0\).}

\textit{Proof.} By Lemma 2, we deduce that the nonzero curvature entries on the vector field (up to the usual symmetries) are given by

\[
R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = -e^{2f(x_1)}((f'(x_1))^2 + f''(x_1)), \quad R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) = e^{4f(x_1)}(f'(x_1))^2
\]

and \(R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = -e^{4f(x_1)}(f'(x_1))^2\),

where \(\{i, j\}\) and \(\{a, b\}\) are unique sets contained in \(U\). Now consider the frame \(\{X_1, X_2, \ldots, X_{2n+1}\}\) (using the same abuse of notation as before) where

\[
X_i = \begin{cases} 
\partial x_i & \text{if } i \equiv 1 \pmod{2}, \\
e^{-2f(x_1)}\partial x_i & \text{if } i \equiv 0 \pmod{2}.
\end{cases}
\]

Noting that if \(\{i, j\}\) \(\in \bar{U}\), then \(i\) is incongruent to \(j\) modulo 2, it easily follows that

\[
g(X_1, X_1) = g(X_2, X_3) = g(X_4, X_5) = \cdots = g(X_{2n}, X_{2n+1}) = 1
\]

and

\[
R(X_1, X_i, X_j, X_1) = -((f'(x_1))^2 + f''(x_1)), \quad R(X_i, X_j, X_j, X_i) = (f'(x_1))^2,
\]

\[
R(X_i, X_a, X_b, X_j) = -f'(x_1)^2,
\]

for any distinct pairs \(\{i, j\}\) and \(\{a, b\}\) in \(\bar{U}\). We already know that \(f'(x_1) = -((f'(x_1))^2 + f''(x_1))\) and also that \(f'\) is surjective onto \(\mathbb{R}^+\). Thus, for any \(t > 0\), there exists a point \(P = (y_1, y_2, \ldots, y_{2n+1}) \in M\) such that \(f'(y_1) = t\), and on the frame \((X_1, X_2, \ldots, X_{2n+1})\), we find that

\[
R(X_1, X_i, X_j, X_1) = t, \quad R(X_i, X_j, X_j, X_i) = t^2,
\]

and

\[
R(X_i, X_a, X_b, X_j) = -t^2.
\]
In other words, for any \( A \in G \), there exists a \( P \in M \) such that \( A^*\mathcal{M} = \mathcal{M}_P \). We conclude that \((M, g)\) is \( G \)-modeled up to order 0 with respect to the model space \( \mathcal{M} \).

We now check that the manifold is not \( HCH_0 \). Calculating the values in (1), through basic counting, one could deduce that

\[
\tau = (4n)t - \left(2n + 8\binom{n}{2}\right) t^2
\]

and that

\[
||R||^2 = 8nt^2 + \left(4n + 16\binom{n}{2}\right) t^4
\]

for any \( t > 0 \). Hence, as a function of \( t \),

\[
F(t) := \frac{\tau^2}{||R||^2} = \frac{(4n^3 - 4n^2 + n)t^2 - (8n^2 - 4n)t + 4n}{(2n - 1)t^2 + 2},
\]

in which case,

\[
F'(t) = \frac{(16n^3 - 16n^2 + 4n)t^2 + (16n^3 - 32n^2 + 12n)t - (16n^2 - 8n)}{(2n - 1)t^2 + 2}.
\]

Since \( F'(t) \neq 0 \), by Corollary 1, we conclude that our manifold is not \( HCH_0 \), and hence not \( CH_0 \) either. \( \square \)

### 3.2 Construction for \( m = 2n + 2 \)

**Lemma 3.** Let \( M = \mathbb{R}^{2n+2} \) with coordinates \((x_1, x_2, \ldots, x_{2n+2})\). Suppose \( f \) is a function of only \( x_1 \). Let \( g \) be a metric on \( M \) with nonzero entries given by

\[
g(\partial x_1, \partial x_1) = 1
\]

and

\[
g(\partial x_2, \partial x_3) = g(\partial x_4, \partial x_5) = \cdots = g(\partial x_{2n}, \partial x_{2n+1}) = g(\partial x_{2n+2}, \partial x_{2n+2}) = e^{2f(x_1)}.
\]

The following hold:

1. The nonzero covariant derivatives of the coordinate frames are

\[
\nabla_{\partial x_1} \partial x_k = \nabla_{\partial x_k} \partial x_1 = f'(x_1)\partial x_k \text{ and } \nabla_{\partial x_i} \partial x_j = \nabla_{\partial x_j} \partial x_i = -f'(x_1)e^{2f(x_1)} \partial x_1
\]

where \( k = 2, 3, \ldots, 2n + 1 \), and \( \{i, j\} \in \hat{U} := \{\{i, j\} : g_{x_i x_j} = g_{x_j x_i} = e^{2f(x_1)}, i \neq j\} \).

2. The nonzero curvature entries up to symmetry are

\[
R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = R(\partial x_1, \partial x_{2n+2}, \partial x_{2n+2}, \partial x_1) = -e^{2f(x_1)}(f'(x_1))^2 + f''(x_1))
\]

and \( R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = R(\partial x_{2n+2}, \partial x_i, \partial x_j, \partial x_{2n+2}) = -e^{4f(x_1)}(f'(x_1))^2 \),

where \( \{i, j\} \) and \( \{a, b\} \) are unique sets contained in \( \hat{U} \).
Proof. The proof of this Lemma is similar to the proof of Lemma 2 and thus is omitted.

Now, for our construction, let our manifold be given by

\[ M := \{(x_1, x_2, \ldots, x_{2n+2}) \in \mathbb{R}^{2n+2} : x_1 > 0\} \]

and let the metric \( g \) on \( M \) be defined as in the previous Lemma. We again suppose that \( f(x_1) = -x_1 + \ln(e^{x_1} - 1) \). Suppose \( \mathcal{M} := (V, \phi, R_0) \) is the model space where \( V = \text{span}\{X_1, X_2, \ldots, X_{2n+2}\} \), the nonzero inner product entries are

\[ \phi(X_1, X_1) = \phi(X_2, X_3) = \cdots = \phi(X_{2n}, X_{2n+1}) = \phi(X_{2n+2}, X_{2n+2}) = 1. \]

Defining \( \hat{U} := \{\{i, j\} : \phi(X_i, X_j) = \phi(X_j, X_i) = 1, i \neq j\} \), we let the nonzero algebraic curvature entries be given by

\[ R_0(X_1, X_i, X_j, X_1) = R_0(X_1X_{2n+2}, X_{2n+2}, X_1) = 1, \quad R(X_i, X_j, X_j, X_i) = 1 \]

and \( R(X_i, X_2, X_b, X_j) = R(X_{2n+2}, X_i, X_j, X_{2n+2}) = -1, \)

where \( \{i, j\} \) and \( \{a, b\} \) are unique sets in \( \hat{U} \). We also let \( G \leq \text{Gl}(V) \) be the set of \((2n+2) \times (2n+2)\) matrices \( A = [a_{ij}] \) such that

\[ a_{ij} = \begin{cases} 1 & \text{if } i = j \equiv 1 \pmod{2}, \\ t & \text{if } i = j \equiv 0 \pmod{2} \text{ and } i \neq 2n + 2, \\ \sqrt{t} & \text{if } i = j = 2n + 2 \\ 0 & \text{otherwise} \end{cases} \]

where \( t \in \mathbb{R}^+ \), so that \( G \cong \mathbb{R}^+ \). We again let \( G \) act on \( \mathcal{M}(V) \) via

\[ G \times \mathcal{M}(V) \rightarrow \mathcal{M}(V) \]

\[ (A, (V, \phi, R_0)) \mapsto (V, \phi, A^*R_0), \]

where \( A^* \) represents precomposition by \( A \).

**Theorem 4.** As defined above, the manifold \((M, g)\) is \( G \)-modeled up to order 0 with respect to the model space \( \mathcal{M} \) and the given group action on \( \mathcal{M}(V) \). In addition, \((M, g)\) is not \( \text{HCH}_0 \).

**Proof.** The details of this proof are similar to that of Theorem 3, and hence we only mention the change of frames. We consider the frame \( \{X_1, X_2, \ldots, X_{2n+2}\} \) (using the same abuse of notation as before) where

\[ X_i = \begin{cases} \partial x_1 & \text{if } i \equiv 1 \pmod{2}, \\ e^{-2f(x_1)} & \text{if } i \equiv 0 \pmod{2} \text{ and } i \neq 2n + 2, \\ e^{-f(x_1)} & \text{if } i = 2n + 2 \end{cases} \]

As before, if we set \( f' = t \), then

\[ R(X_1, X_i, X_j, X_1) = R(X_1, X_{2n+2}, X_{2n+2}, X_1) = t, \quad R(X_i, X_j, X_j, X_i) = t^2. \]
and \( R(X_i, X_a, X_b, X_j) = R(X_{2n+2}, X_i, X_j, X_{2n+2}) = -t^2 \),

where \( \{i, j\} \) and \( \{a, b\} \) are unique sets contained in \( \hat{U} \).

Again, as in Theorem 3, it can be verified that the value \( \tau^2 / \|R\|^2 \) is non-constant, and using Corollary 1, we see that \( (M, g) \) is neither \( CH_0 \) nor \( HCH_0 \).

The combination of Theorems 3 and 4 proves Theorem 2.

4 Conclusion and Open Problems

While the work above tampers with the dimension of a \( G \)-modeled manifold, with the dimension of \( G \) being 1, it was shown in [3] that there is a 3-dimensional manifold that is \( G \)-modeled with respect to a model \( \mathcal{M} \), where \( G \) had dimension 2. In fact, in this example, the manifold was \( G \)-modeled up to order 1. The next steps would be to investigate manifolds that are \( G \)-modeled up to order \( k \geq 1 \), where the Lie group has dimension \( l \geq 1 \). In addition, one might investigate more topologically interesting groups, such as a compact Lie group.

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References

