Nested Links, Linking Matrices, and Crushtaceans

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Abstract

If two knots have homeomorphic complements, then they are isotopic; however, this is not true for links. Geometry and topology can be combined to show when certain hyperbolic links have homeomorphic complements. What remains is to determine if the links themselves are isotopic. We determine an algorithm to find the linking numbers of two types of hyperbolic links known as fully augmented links (FALs) and nested links from their respective graphical representations, known as crushtaceans. We will show that some links constructed from the same crushtacean are not isotopic. The algorithm shows that topological information can be obtained directly from the crushtacean’s combinatorial data. One useful application of the algorithm is to distinguish links with homeomorphic complements.

1 Introduction & Background

A knot is a closed loop that exists in $S^3$. One way to imagine this is to tie a knot in a piece of string and then glue the ends together. This is a knot because there is no way to untangle it. An unknot, also known as a trivial knot is a circle. A knot contains one component while a link contains more than one component [1].

![Figure 1: Glue both ends to form a knot on the lefthand image. The righthand image is an unknot.](image)

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![Figure 2: A Hopf link contains two knot components.](image)

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The signed number of times a component links with another is known as a linking number. The orientation of knot circles in a link determines if the linking number is positive or negative. To see what constitutes a
positive or negative crossing, refer to Figure 3. Linking numbers are useful in determining whether two links are distinct, and more efficient than using Reidemeister moves. Reidemeister moves allow the projection of the knot to be changed which leaves the linking number unchanged.

Figure 3: A positive orientation means that the understrand will point in the same direction as the overstrand when rotated clockwise. For a negative orientation, this direction is counterclockwise [1].

Let \( L \) be a link that contains knot components \( J \) and \( K \). The linking number between \( J \) and \( K \) is defined as the sum of positive and negative crossings divided by two. \( L(J,K) = \frac{1}{2} \sum \epsilon(c) \) where \( c \) is each crossing and \( \epsilon(c) \) the sign at each crossing. The sum is taken over crossings involving strands from both components \( J \) and \( K \). (For more details see [1]).

**Example 1.1. An example of the linking numbers for a Hopf link.**

![Figure 4: With the chosen orientation for the Hopf link, the linking number is \(-1\).](image)

\[ L(J,K) = \frac{1}{2} \sum \epsilon(c) \]

1.1 Fully Augmented Links & Nested Links

A link complement is the space surrounding the link. A link is hyperbolic if its complement admits a hyperbolic metric [2]. Fully augmented links (FALs) and nested links are two types of hyperbolic links that can be formed from link diagrams. Let \( L \) be a link with alternating crossings in each twist region. To form a FAL, place an unknotted component, known as a crossing circle, around each twist region. Then, remove all full twists such that only half-twists or no twists remain. If the number of twists in the region is odd, the result is a half twist, and if there are an even number of twists, then no twists remain. In FALs, each crossing circle is punctured twice by one or two knot circles [3]. An example of an FAL formed from a link can be seen in Figure 5.

Nested links are formed in the same manner; however, more than two knot circles can exist. By drawing multiple crossing circles nested inside each other, each crossing circle is still only punctured twice [4]. Please refer to Figure 6 for an example.
1.2 Cell Decomposition

We can determine the shape of the hyperbolic link complement through a process called **cell decomposition** that takes place in $S^3$. Because the complement is a complete hyperbolic manifold, it can be thought of as different polyhedra glued together in accordance with the structure of the manifold. This gluing is known as a **gluing pattern**. The complement has 0, 1, and 2-cells. 0-cells correspond to knot circles and 1-cells are intersections of 2-cells, which are crossing circles and the projection plane. The first step of the process is to cut along the planar 2-cells, splitting them into two isometric regions. This step is known as “the pita bread slice,” or “butterflied.” One can think of it as slicing the crossing circle and then flattening each half. The next step is to “pinch” the crossing circles along their diameter such that the diameter has shrunken to an ideal point. Next, shrink the 1-cells (shown as black edges in Figure 7). We now have the link complement.

Now, we can obtain the graphical representation of a FAL or nested link, known as the crushtacean. To determine the crushtacean, place a vertex in the center of each shaded region. Connect vertices on the boundary of the face to the vertex in the center, painting the edge in accordance with the vertices on the boundary [5].
A crushtacean is a simple, trivalent, planar graph that contains painted and unpainted edges [6]. Painted edges are drawn in color and unpainted edges are drawn in black. Painted edges of a crushtacean correspond to crossing circles and unpainted edges correspond to knot circles. Painted edges are either in the form of perfect matchings, balanced trees, or balanced forests, which are described in the next subsection. An example of a painted crushtacean can be seen in Figure 8.

![Crushtacean Example](Image)

Figure 8: A crushtacean with painted and unpainted edges.

1.3 Balanced Trees

An edge symmetric spanning tree [3], which we will call a balanced tree is a tree that admits an involution, $\varphi$ that fixes an edge, $\beta$. The edge $\beta$ such that $\varphi(\beta) = \beta$ will be called the balanced edge. The vertices $v$ and $\varphi(v)$ will be called mirrored vertices. An example of a tree with these labelings can be found in Figure 9. The tree excluding the balanced edge, $(t \setminus \beta)$, contains two isomorphic subtrees denoted $\tau$ and $\varphi(\tau)$ where if $v$ and $\varepsilon$ exist in $\tau$ then $\varphi(v)$ and $\varphi(\varepsilon)$ exist in $\varphi(\tau)$. Each endpoint of the balanced edge is considered to be the root of its corresponding subtree. Notice that the painted edges of Figure 8 form a balanced tree.
Figure 9: $\varphi(v)$ and $v$ are mirrored vertices. $\beta$ is the balanced edge. Each pair of colors corresponds to a pair of mirrored vertices.

If all the trees in the crushtacean are of length one, this is known as a perfect matching. In perfect matchings, vertices of painted edges do not connect to other painted edges. If there are multiple balanced trees, or a combination of balanced trees and perfect matchings, this is known as a balanced forest. A balanced forest can be seen in the crushtacean from Figure 7. Perfect matchings coincide with FALs, and balanced tree and balanced forests coincide with nested links.

Figure 10: A crushtacean with a balanced tree. The red edge is the balanced edge. The pairs of colored vertices are mirrored vertices.

2 Proofs for FALs and Nested Links

We will prove that the number of crossing circles for FALs and nested links can be determined directly from the crushtacean and find a correspondence between crossing circles and painted edges. We will also prove that the number of knot circles is at most the number of crossing circles.

Theorem 2.1. The number of crossing circles is $\frac{v}{2}$ in nested links and FALs.

Proof. Suppose $\Gamma$ is a crushtacean with a total of $v$ vertices and $E$ edges, where $p$ is the number of painted edges and $u$ is the number of unpainted edges. Since FALs can correspond to perfect matchings and nested links can correspond to balanced trees and forests, let us divide the proof into two parts.

Let $\Gamma$ be a crushtacean with a perfect matching, and let $c$ be the number of crossing circles for $\Gamma$. Because of the trivalent nature of crushtaceans, each vertex has three edges. In a perfect matching, painted edges do not share vertices, so each vertex has exactly one painted edge. Because painted edges correspond to crossing circles, this means $p = c$ and $c = \frac{1}{3}E$. In a finite graph, the sum of degrees is equal to twice the number of edges. Since the crushtacean is trivalent, there will be 3 edges per vertex, so $E = \frac{3v}{2}$. Using substitution, we can see that $c = \frac{v}{2}$, and $u = v$. 
Now suppose $\Gamma$ has a balanced forest corresponding to nested links. The coloring of a balanced tree is as follows: the balanced edge, $\beta$ has its own color. Every edge $\epsilon \neq \beta$ and its mirrored edge, $\varphi(\epsilon)$ have the same color.

In a tree, $t$ with $v_t$ vertices and $E_t$ edges, $E_t = v_t - 1$, so $t$ has $v_t - 1$ edges. The balanced edge corresponds to one crossing circle, so there are $v_t - 2$ edges left in the tree. (This means the number of painted edges for a single tree is $v_t - 2$). Thus, there are $\frac{v_t - 2}{2}$ more crossing circles, so in total, $c = 1 + \frac{v_t - 2}{2} = \frac{v_t}{2}$.

For a balanced forest, let $T$ be the total number of trees. Each tree has $v_t - 1$ edges, so there are $v - T$ painted edges for the forest. Since each balanced edge in a tree corresponds to 1 crossing circle, there are $v_t - 2$ edges left in each tree, and $v - 2T$ edges left in the forest. For the total number of crossing circles, $c = T + \frac{v - 2T}{2} = \frac{v}{2}$.

Since $c = \frac{v}{2}$ for perfect matches, balanced trees, and balanced forests, the number of crossing circles is the same for nested links and FALs.

**Corollary 2.1.1.** The number of knot circles is less than or equal to the number of crossing circles for FALs and nested links.

**Proof.** Suppose $\Gamma$ is a crushtacean with $v$ vertices and $E$ edges. Let $p$ be the number of painted edges, $u$ be the number of unpainted edges, $c$ be the number of crossing circles, and $k$ be the number of knot circles.

Suppose $\Gamma$ is a crushtacean with a perfect matching. A single knot circle uses at least two edges because in order for a knot circle to use only one edge, its endpoints must be mirrored vertices. In a perfect matching, this would result in a multiedge; however, crushtaceans do not contain multiedges. This means a knot circle must use at least two edges in a perfect matching. Thus, the maximum number of knot circles for a single crushtacean is $\frac{v}{2}$. Because $c = \frac{v}{2}$, the number of knot circles, $k$ cannot be greater than the number of crossing circles. It is possible for $k < c$, so $k$ must be less than or equal to $c$.

Now, suppose the crushtacean is spanned by a balanced forest, $F$ with $T$ trees. Let $\ell$ be a leaf with two unpainted edges. Only one of the unpainted edges can connect to the pair of mirrored vertices, since both connecting would result in a multiedge. This means in a leaf, there is at most one edge that connects two mirrored vertices. The other edge therefore must be part of a knot circle with at least two knot edges. Let $d = u - k$ be the difference between the number of unpainted edges and the number of knot circles. For the maximal case, where each mirrored vertex is connected by one knot edge $d \geq 1$, so $k = u - d \leq \frac{v}{2} + 1 - 1 \leq \frac{v}{2}$. For forests, this would be extended to $k \leq \frac{v}{2} + T - T \leq \frac{v}{2}$. Thus, since $c = \frac{v}{2}$, this means $k \leq c$.

## 3 Linking Matrix Algorithm (LMA) Determines the Linking Number from the Crushtacean

In this section, we will introduce the linking matrix for FALs and nested links and determine an algorithm for computing it directly from the crushtacean. We will also use the linking matrix to distinguish links with homeomorphic complements.

### 3.1 Construct Links Directly from the Crushtacean

To construct the nested link directly from the crushtacean, utilize the following rotation system to determine if crossing circles are flat or twisted.

1. Mark all pairs of mirrored vertices.

2. Starting with any edge from a pair of mirrored vertices, keep track of the counterclockwise rotation order of the edges.
3. Start with that same edge for the other mirrored vertex. If the counterclockwise rotations have the same order of edges, then the knot circles are twisted. If the orders are opposite, then the knot circles are flat.

Figure 11: The counterclockwise rotation for the pair of mirrored vertices of the balanced edge is opposite, so the link is flat. This is true for all the pairs of mirrored vertices in this particular crushtacean.

Figure 12: An example of a balanced edge whose mirrored vertices have the same counterclockwise rotation which will result in a twisted link.

**Theorem 3.1.** If the counterclockwise rotation order of edges connecting a pair of mirrored vertices is the same, the knot circles are twisted. If it is opposite, the knot circles are flat.

**Proof.** $v$ and $\varphi(v)$ share the same color edges since their respective subtrees are isomorphic. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the edges corresponding to $v$ and $\varphi(\varepsilon_1), \varphi(\varepsilon_2), \varphi(\varepsilon_3)$ be the edges corresponding to $\varphi(v)$. Let us look at the counterclockwise order of edges for $v$. Beginning with a single edge, there are two possible orderings for the other two edges. Without loss of generality, let us begin with $\varepsilon_1$. The order of the other two edges for $v$ can either be $\varepsilon_2, \varepsilon_3$ or $\varepsilon_3, \varepsilon_2$. If the counterclockwise order of edges for $\varphi(v)$ are the same as $v$, this means those painted edges in tree are twisted. A reflection across the balanced edge (i.e. “gluing” the mirrored vertices of the balanced edge) would cause two distinctly painted edges to share an edge. Because both subtrees are isomorphic, the tree would need to twist in order for $\varphi$ to map vertices and edges to their mirrors.

If the counterclockwise edge orderings for $v$ and $\varphi(v)$ are opposite, a twist does not need to occur for the subtrees to be isomorphic since two painted edges of the same color share an edge when reflected across the balanced edge. Thus, $v$ and $\varphi(v)$ would be flat.

**Example 3.1.** The balanced edge will always correspond to the outermost crossing circle in a nesting. Let the “inner square” be $\tau$ and the “outer square” be $\varphi(\tau)$. We can consider $\tau$ to be the “front” of the link which means the purple edge is on the right side of the balanced edge, and the blue edge is on the left. Looking at the endpoints of the blue edges, shown as pink vertices, let us apply the counterclockwise rotation system. Starting with the blue edge of $\tau$, the rotation order is blue, an unpainted edge, and green. Looking at the blue edge of $\varphi(\tau)$, the rotation order is blue, green, and an unpainted edge. Thus, the knot edge connecting the blue edges of $\tau$ and $\varphi$ is flat. Because the endpoints of the purple and green edges each have two unpainted edges, we can choose whether those knot edges are flat or twisted. Let us have them be flat. The corresponding link can be seen in Figure 14.

**Figure 13:** Oriented subtrees with directed knot edges.

**Figure 14:** The corresponding nested link drawn from the crushtacean in Figure 13.
3.2 Orienting the Crushtacean

Let $\Gamma$ be a crushtacean containing a balanced forest. Let $t$ be a balanced tree in the crushtacean. Orient the balanced tree by choosing which subtree to call $\tau$ and $\varphi(\tau)$. Recall that $\varphi^2(\tau) = \tau$ is an involution, where $\tau$ contains the preimage of edges and vertices and $\varphi(\tau)$ contains the image. An example of an oriented crushtacean with one balanced tree is shown in Figure 15.

![Figure 15: The vertices corresponding to the balanced edge, shown in orange are a pair of mirrored vertices as well as the pink, dark blue, and yellow vertices. The crushtacean is oriented such that knot edges entering the side of the balanced edge closer to the inner square first enter $\tau$.](image)

Now, orient a knot edge. The knot circle passes through crossing circles represented by edges on the path between mirrored vertices, where it enters and leaves the tree. If the path of a knot edge goes from $\tau$ to $\varphi(\tau)$, then it is linked +1 times. If the knot edge goes from $\varphi(\tau)$ to $\tau$, then it is linked −1 times.

![Figure 16: Directed knot edges. The knot circles go through one subtree, and then through the balanced edge to connect the mirrored edges.](image)

3.2.1 Constructing the Matrix

We will determine the linking number based on our choices of orientations. Recall that because two subtrees in a balanced tree are isometric, two painted edges of the same color correspond to one crossing circle. Summing each linking number between the crossing circle and knot circle gives the net times the crossing circle is linked. Here is an example of the linking matrix for Figure 16. The rows refer to knot circles and the columns refer to crossing circles.
Theorem 3.2. For FALs, half-twists can be chosen so that knot circles do not link each other. Thus, knot circles linking other knot circles need not be considered in the linking matrix.

Proof. By definition, crossing circles do not link other crossing circles, so they do not need to be included in the rows. While it is possible for a knot circle to link another knot circle, this can be simplified to being unlinked. We can see from the Dehn twist that half twists can be chosen such that the separate knot components are unlinked. In other words, we can use the Dehn twist to invert the crossing, thereby changing the linking number by -1.

Figure 17: The Dehn twist is done on the two strands in the top half. Since there is a twist between the bottom two strands, this is isotopic to "sliding the crossing disk down" so that the bottom twist is in the top. A Reidemeister II move\(^2\) is used to remove the unnecessary twist leaving us with the opposite crossing.

For FALs, a crossing only occurs at the crossing circle. As shown in the diagram, since a crossing can be changed from under to over or vice versa, this means that a crossing can be changed in an FAL such that it is connected to the same knot circle.

Example 3.2. The twisted Borromean rings are an example of a FAL whose crossings can be changed via the Dehn twist.

\(^2\)Reidemeister II moves allow for the addition or removal of two crossings [1].
Figure 18: Using the Dehn twist, the linking number in the twisted Borromean rings is changed so that the knot circles are not linked.

### 3.3 Links with Homeomorphic Complements are not Isotopic

We will prove that linking matrices for isotopic links can be related by a sequence of moves. We will also show that links constructed from the same crushtacean are not isotopic.

**Theorem 3.3.** Suppose $L$ and $L'$ are isotopic links. In other words, there is a homeomorphism $h : S^3 \rightarrow S^3$ with $h(L) = L'$. If $M$ and $M'$ are their linking matrices, then, $M$ and $M'$ are related by a sequence of the following operations:

1. Multiply any row or column by $-1$.
2. Interchange any two rows or columns.

**Proof.** The direction of the knot edge can be chosen to go from $\tau$ to $\varphi(\tau)$ or $\varphi'((\tau)$, or vice versa. This determines if the knot circle is oriented counterclockwise or clockwise, and determines if the linking numbers with the crossing circles it passes through are $-1$ or $1$. If the orientation of the knot circle is reversed, then the linking numbers for each crossing circle it passes through are also reversed. Thus, we can multiply each row of the linking matrix by $-1$ to represent reversed knot circle directions. If there are $k$ rows in the matrix, then there are $2^k$ possible ways to write the rows of the matrix since there are two ways to express each row.

The order of rows and columns can be different since the actual linking number between a knot circle and crossing circle remains unchanged. Thus, such a reordering of rows and columns corresponds to a relabeling of link components. For nested links and FALs, crossing circles cannot link other crossing circles. Thus, each row represents a knot circle. For nested links, since it has not yet been proven that twisted knot circles can always be unlinked, knot circles must also be considered in the columns of the matrix.

**Corollary 3.3.1.** If $M$ and $M'$ have a different number of zeros, then $L$ and $L'$ are not isotopic.

**Proof.** This is because each 0 remains unchanged by the moves described in **Theorem 3.3.** There is no way to get from 0 to 1 or -1.

**Example 3.3.** The sequence of operations show that link complements can be homeomorphic while the links themselves are not isotopic.

Using the LMA, we can observe the following linking matrices:
Figure 19: Depicted are two balanced trees who exhibit the same gluing pattern since the same edges are glued along the mirrored vertices. This means their link complements will be homeomorphic.

From the sequence of operations, we see that the number of zeroes in both matrices are different, hence their links cannot be isotopic.

4 Conclusion

We were able to determine an algorithm from the crushtacean to show that links with homeomorphic complements are not isotopic. This algorithm also obtains topological information directly from the combinatorial data.

5 Open Questions

1. Consider a crossing circle in a flat FAL. If it is twice punctured by the same knot circle, then it is unlinked. For twisted FALs, if a crossing circle is twice punctured by the same knot circle, it can be linked ±2 times. How can this be used to simplify the linking matrix for an FAL?

2. Because crossing circles in FALs are punctured exactly twice, are there stricter equivalence relations for distinguishing these types of links?

3. Can twisted knot circles in nested links be simplified such that they are unlinked?

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References


