Knots in a Cubic Lattice

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Abstract

In this paper, we discuss the composition of knots on the cubic lattice. One main theorem deals with finding a better upper bound for the stick number of a composed knot in relation to the stick number of the original two knots. Other theorems go into detail about the composition of two or more trefoil knots and two or more figure-eight knots, finding upper bounds for stick numbers.

Keywords : cubic lattice, lattice polygon, composition, stick number on the cubic lattice.

1 Introduction

A knot is a spatial embedding of S^1 in S^3. Stick knots are made with only sticks and no curves. The *cubic lattice* is the graph in R^3 whose vertices consist of all points with integer coordinates and whose edges consist of line segments of unit length connecting these vertices. A *lattice polygon*, or a knot, on the simple cubic lattice is a piecewise linear, simple closed curve. Further, each linear piece of this polygon is a unit length segment with its endpoints on the cubic lattice.

Now, define the *stick number* s(K) of a knot K to be the least number of straight sticks necessary to make K see[1]. Then, the *stick number* in the cubic lattice, $e_{\perp}(K)$, is the fewest number of sticks required to build a knot, K, in the cubic lattice. Thus, because of the restrictions that the cubic lattice yields upon the structure of a knot, stick numbers tend to increase dramatically when converting to cubic lattice form.

Moreover, there are several known results within this field, although many open questions remain. For example, in 1992, it was found that the minimal stick number for a nontrivial cubic lattice knot is 24, compared to only 6 sticks for a general stick knot[3]. The knot that was found is a three-crossing knot, the most basic knot. Although this result may seem trivial, we have yet to prove the stick numbers of any other specific knots, such as those with four crossings, five crossings, or seven crossings. Using a computer program, the authors of[4].

it has been conjectured that the figure-eight knot has a minimum of 30 sticks, the 5_1 knot has 34 sticks, the 5_2 knot has 36 sticks, and the 6_1, 6_2 and 6_3 knots all can be constructed with a minimum of 40 sticks in the cubic lattice, yet this is still to be proven. Furthermore, it has been shown that the compostition of two cubic lattice knots yields no more than the original amount of sticks, or that $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2)$ (see[2]). So, open questions remain to the existence of a better upper bound for composition and to finding the stick numbers of various knots on the cubic lattice.

Most importantly, these knots are of particular importance in chemistry in simulating various circular molecules, especially the ones with relatively big volumes.[3]

2 General Definitions

First, we define several terms that will be relevant to the whole paper.

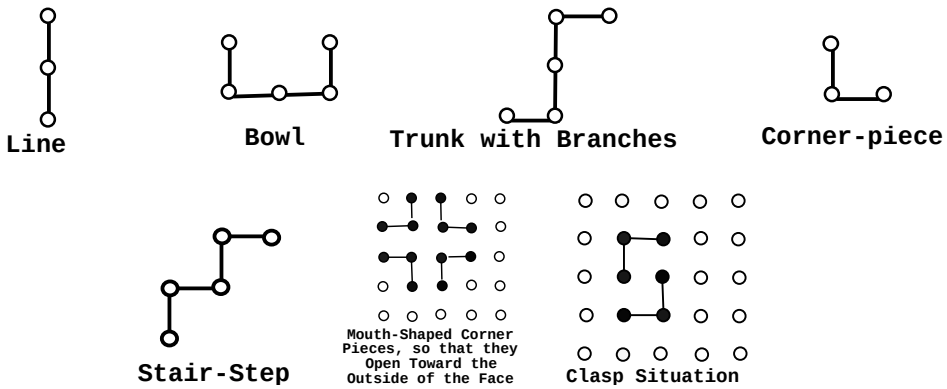
Definition 1: Each unit segment in a polygon is called a *step*.

Definition 2: The endpoints of the steps are called *vertices*.

Definition 2: A *lattice polygon* P (knot) is called reducible if there is a shorter polygon which is topologically equivalent.

***Whenever we refer to a *polygon* in this paper, it will always mean a polygon on the cubic lattice.

Further, we will refer to certain shapes as illustrated below:



Common Shapes Throughout the Paper
Figure 2.1

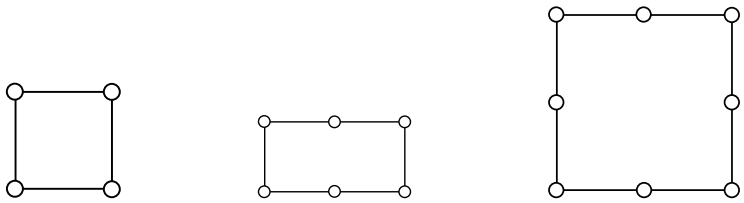
3 The Stick Number of the Hopf Link in a Cubic Lattice

Here, we are concerned with the minimal amount of sticks needed to construct the Hopf Link in the restricted cubic lattice.

THEOREM 3.1: The minimal stick number of the Hopf link in the cubic lattice is 16.

PROOF:

The *Hopf link*, by definition, is two unknots linked with two crossings[1]. When working under the restrictions of the cubic lattice, we find that each knot must have an even number of sticks because it must have four sides in order to contain only ninety degree angles. Thus, the smallest number of sticks that one unknot could have would be 4. Therefore, in proving the theorem, we study unknots with 4 sticks, 6 sticks, and those with 8 sticks, as pictured below.

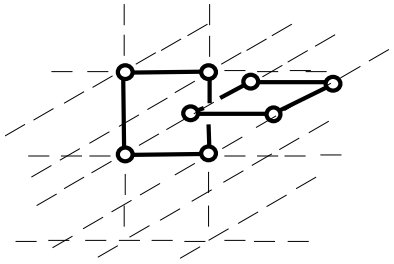


(a) Four Sticks (b) Six Sticks (c) Eight Sticks

Figure 3.1

We notice that there exist six possible combinations for the linking of the Hopf Link using the three dimensions of unknots listed above. We can link an unknot of 4 sticks with one of 4 sticks, one of 4 with one of 6, 4 to 8, 6 to 6, 6 to 8, and 8 to 8. In order for the two unknots to link, they must be oriented so that both unknots are in perpendicular planes and are situated along the cubic lattice.

Thus, we notice that in the situation of linking 4 sticks with 4 sticks, the unknots cannot intersect along the structure of the cubic lattice, as pictured below.



The flat unknot is not on the cubic lattice grid.

Figure 3.2

Obviously, this will occur whenever an unknot has a side of length 1 because anything linked through that edge cannot possibly be aligned on the cubic lattice. Therefore, it is impossible to have a correct link on the cubic lattice using the unknots of stick numbers 4 and 6. This then leaves us only with the possibility of using the two unknots of stick number 8 to create the Hopf Link. Thus, we are left with a minimal stick number of 16 for the Hopf Link in the cubic lattice. □

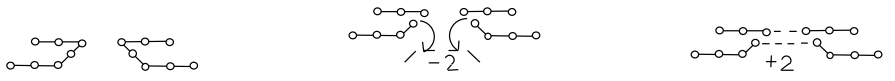
4 Composing on the Cubic Lattice

4.1 Definitions

Before we begin working on the theorem, we will define a few terms:

Definition 1: Given two projections of knots, we can define a new knot obtained by removing a small arc from each knot projection and then connecting the four endpoints by two new arcs. We call the resulting knot the *composition* of the two knots[1].

Composition of two knots on the cubic lattice occurs by joining the two structures without tampering with the original structure of the knot. A basic way of doing this involves erasing one stick from the outer edge of each knot. Then, to connect the two, one can add two more sticks to "bridge" the two knots together. This is shown below.

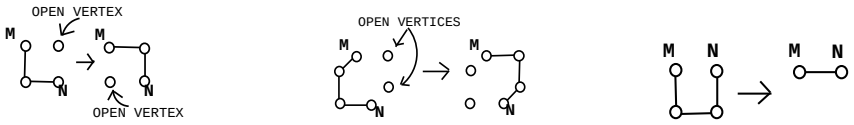


(a) The outer edges of two knots (b) Losing two sticks (c) Final Composition

Figure 4.1.1

Note that this proves that $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2)$ (as in[2]) since adding in two sticks is negated by subtracting two more sticks.

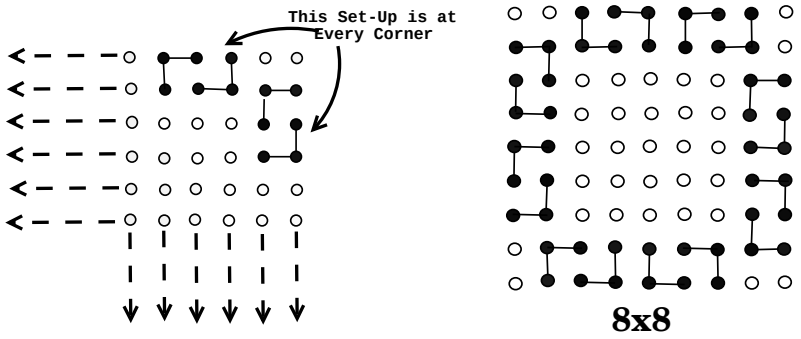
Definition 2: Whenever working with knots on the cubic lattice, one must always find the shortest path from a point M to another point N. Whenever the path can be reduced or changed without changing the knot type of the knot, then we say we can perform a *square move*. Several square moves are shown below[3].



(a) (b) (c)

Figure 4.1.2

Definition 3: A knot is *bad* if every minimal cubic lattice representation has this configuration on all its faces. A knot that is *good* does not have this configuration on all its faces.



The configuration on the face of a bad knot

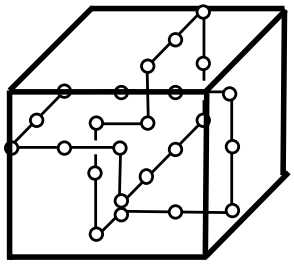
Figure 4.1.3

4.2 The Main Theorem and Some Preparations for its Proof

THEOREM 4.1: $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$ whenever both K_1 and K_2 are Good.

PROOF:

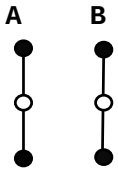
We start out with two knots, Knot A and Knot B. Because each knot is situated on such a rigid structure, we can place each knot within a large rectangular box and shrink the box until at least one edge from the knot lies on each of the six faces.



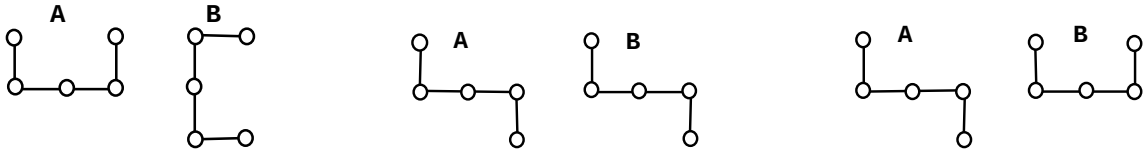
The trefoil knot surrounded by a box so that each face of the box contains edges from the knot

Figure 4.2.1

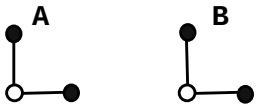
Thus, when composing one knot with another, we will only work with the faces of the boxes. So, we have found four general cases of certain face designs that can exist as shown below.



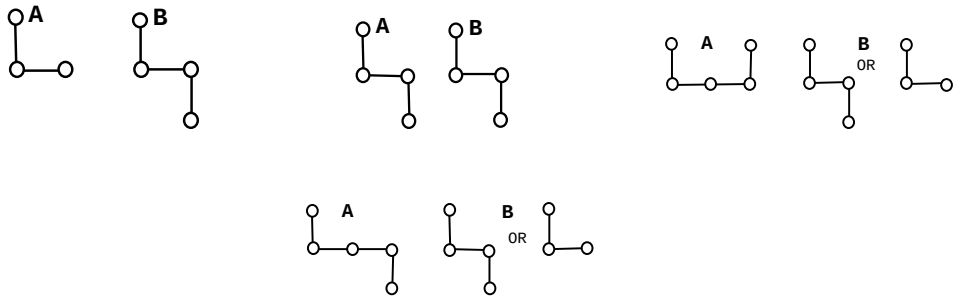
CASE 1



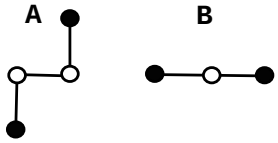
Case 1 Examples



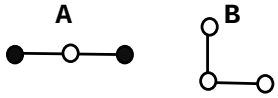
CASE 2



Case 2 Examples



CASE 3



CASE 4

Figure 4.2.2

The Filled in dots represent vertices at which additional sticks may be added, as in the pictured examples. Adding these steps does nothing to hinder the composition of the two knots since we will be focusing on the main structure.

For all these cases, we imagine a cube A with only such a design on all the faces and a cube B with only that design on all of its faces. And, when considering square moves, if we look at situation (c) in Figure 4.1.2, we know that a single line will never exist on the face of the cube. This is illustrated below.

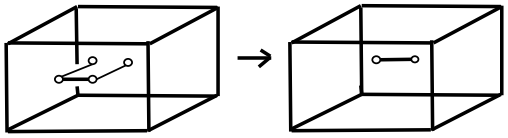


Figure 4.2.3

So, for Case 1, we deal with two cubes that have only "lines" of at least length two on each face. If one face of cube A has a vertical stick of length two and cube B also has a vertical line of length two, then we can easily overlap the two lines, add two steps to bridge the knots together, and then get rid of the original four sticks. Thus, we end up with two less sticks than we began with (Figure 4.2.4).

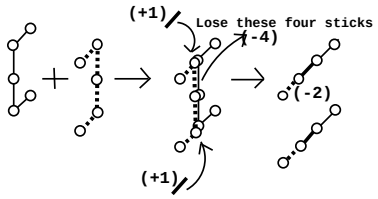


Illustration of Case 1 Scenario

Figure 4.2.4

In the situation when one stick is vertical and the other horizontal, we can easily rotate one cube to align the two sticks. And, we find that if we compose two sticks of different lengths, we will always lose at least two sticks in the end because the original stick must be at least of length two.

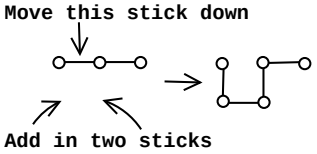
In the examples for Case 1, all that we are concerned with is the base line of at least two steps. Then, no matter what is stemming off the base line, we just overlap the base lines from both knots.

In Case 2, we can easily overlap the two "corner-peices" so that we lose at least four sticks and then add two more for a total of at least two lost. Since a corner-piece can rotate to always overlap some part of the another corner-piece, stair-step, bowl, or trunk we are guaranteed this situation.

In the situation where we have two stair-steps, we need not be concerned that the two might overlap completely, but instead that parts of each overlap. Thus, in the end, all we need are two overlapping corner-pieces to obtain our goal of a loss of two sticks which can easily be done by rotating one knot to get the desired orientation. The same goes when composing a bowl with a stair-step

or a trunk with a stair-step.

For Case 3, we have a new "stair-step" shape that can contain as many steps that will fit on the face. For this case, we must keep in mind that all that exists on cube A are the stair-step pieces (of at least 3 sticks) and all that exists on cube B are the long line pieces. At first it seems that the two cannot line up, but it ends up that we can add two sticks to the picture of cube B to get a "spoon-like" design.



Changing around Cube B of Case 3.2

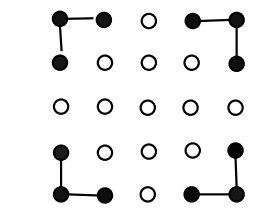
Figure 4.2.5

We will always be able to do this for one of two reasons. In the first one, open vertices already exist to make this addition possible. If these vertices do not exist, it means that those vertices must be occupied by another stick. Repeating this process, we eventually get a stick at the very edge of the face of the cube, and, here, one can add the extra sticks off the face of the cube with no interference (since nothing exists there). And, once we add in the extra two sticks, we will be able to overlap at least six of the sticks to end up with a loss of two (losing six, adding two originally, and adding two more to bridge the knots).

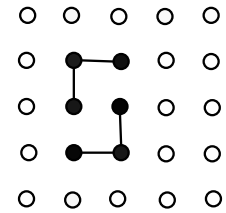
Case 4, on the other hand, is much more complicated. Notice that if we use the previous method of overlapping sticks, removing those that are overlapping, and then adding two sticks to bridge the knots, the best that we can do is end up with the same number of sticks that we began with, the exact conclusion that we are trying to improve on. This happens because the corner-piece has only side length of one step, allowing us to only overlap one step from the corner-piece with one step from the line.

Thus, two possible solutions to the problem arise. The first is to prove that it is not possible for a knot to have only corner-pieces on the faces of its box. Then, we would not even have to worry about this case. This is much more easily said than done, and thus remains an open question: is it possible for a cubic lattic knot to have only corner-pieces on the face of the box which surrounds it?

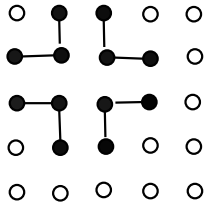
The second solution would be to prove that we can compose these two types of knots without having to add in two more sticks to bridge them. Then, in this case, several scenarios arise, illustrated below.



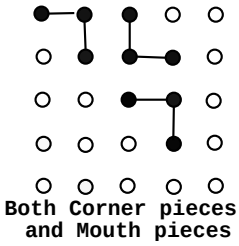
Scenario 1



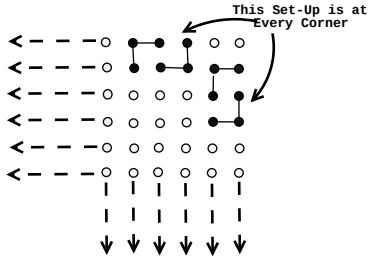
Scenario 2



Scenario 3



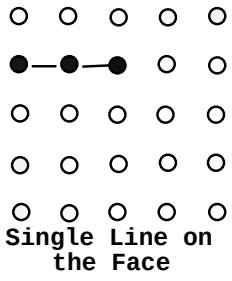
Scenario 4



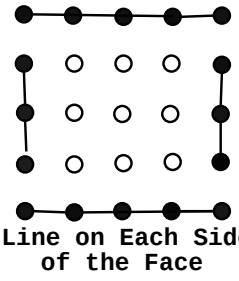
Scenario 5

Possible Situations that can Occur

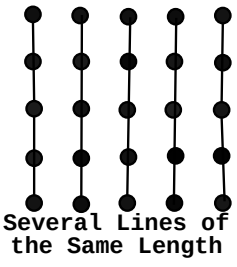
on the Cube with Only Corner Pieces



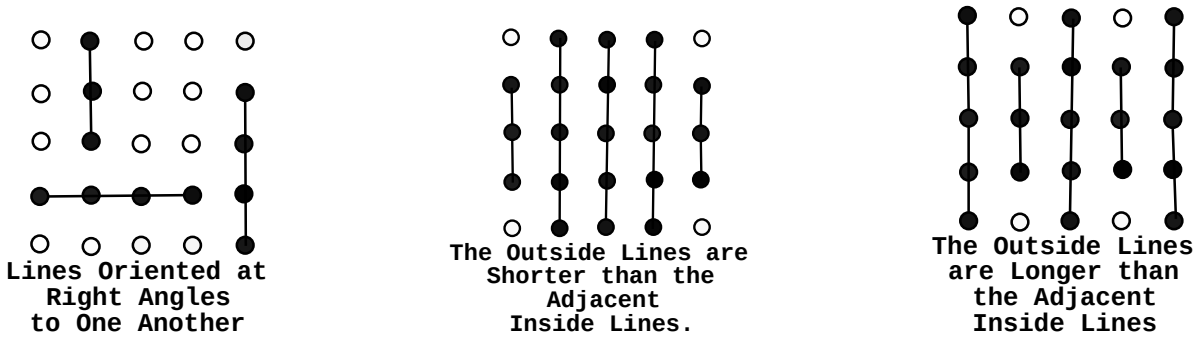
Scenario A



Scenario B



Scenario C



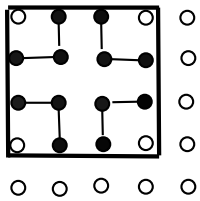
Scenario D Scenario E Scenario F

Possible Situations that can Occur

on the Cube with Only Line Pieces

Figure 4.2.6

Now, we take each face of the box surrounding the cube and draw a rectangular box around everything on the face, creating the smallest rectangle possible. We call this new region the *interior region*, as shown below.



The Interior Region on the Face of a Box Surrounding a Cubic Lattice Knot

Figure 4.2.7

Lemma 4.1: $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$ is true whenever the interior region includes all of the face of at least one face of the box for the bad knot.

So, we are looking at the case in which the interior region *is* the face of the box (that is, it is the same size as the face of the box). Then, we know that the steps are on or near the outer edge of the face of the box.

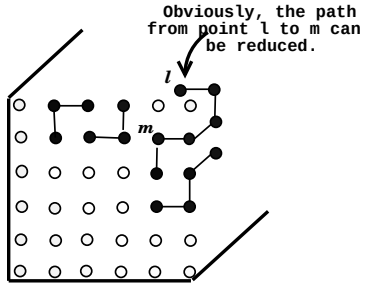
Thus, for Scenario 1, the corner piece(s) are on the outer edge of the box. Then, we can compose anything with one of these corner pieces since there exists a mass of open space surrouding the corner where we do not have to worry about the second knot colliding with the first knot. Therefore, we do not have to bother adding in the two extra bridging steps. This can also be applied to

Scenario 2.

Then, for Scenario 3, since the outer corner vertices are free wherever a mouth piece exists, we can easily square move to create Scenario 1, which is given.

In Scenario 4, although we have both mouth pieces and regular corner pieces, we will always have at least one corner piece or a mouth piece that can square move into a corner piece. Then, although the face may seem cluttered, since the mass of open space exists, we can easily compose without the two knots colliding.

Scenario 5 looks like a more troubling case, yet if the interior region is the the face of the box, then we find that this scenario cannot exist. Because all the faces of the cube have corner pieces (because otherwise we would not be in this mess), we are very restricted as to the direction in which the steps have to go. Figure below show that if the steps follow the only path possible, we are left with a situation which is not reduced. And, since we assume that the knot is in reduced form, we arrive at a contradiction. Thus, Scenario 5 cannot exist when the interior region is the same size as the face of the box.

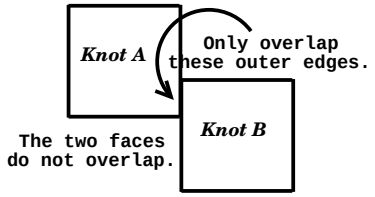


Scenerio 5 Cannot Exist on the Edge of the Box

Figure 4.2.8

Moving on to the Scenarios dealing with the lines, we see that Scenario A could not even exist on all the edges of the face because then the knot would be two dimensional.

For Scenario B, we can compose with any of the corner piece scenarios. All we need to do this is a step on the outer edge of the corner piece knot and this will either exist or we can use a square move to create it. Then, we overlap a step from the outer corner of the interior region (of the knot in Scenario B) with a step also from the outer corner of the interior region of the corner piece knot. That way, the rest of the faces from the knots will not overlap, as illustrated below. Then, Scenario C and F can be done in the same way.



Composing A Scenario B Knot with any Corner Piece Knot

Figure 4.2.9

Scenario D looks trickier, yet since the lines are always of at least length two, there will always be a "buffer" region so that the knots will not overlap. Thus, we just take a step that is furthest from the edge and compose it with a corner piece.

With Scenario E, we have to be more careful that the inner lines do not accidentally overlap with parts of the corner piece knot. But, since it is on the edge of the box, we have nothing to worry about since there exists the open space around it. The only scenario that would not work, would be scenario 5, but since it does not exist when the interior region is at the edges of all the faces, we can ignore it.

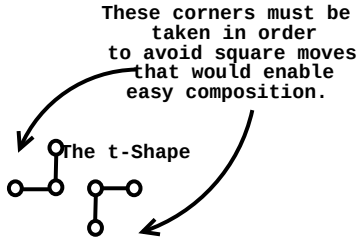
Therefore, we can conclude that when the interior region includes all of the face of at least one face of the box for both knots, case 4 can be composed. Therefore,

$e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$ holds true in this situation. $\square$

Corollary 4.1: $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$ is true when composing two knots, both of which are smaller in dimension than $10 \times 10 \times 10$ sticks.

Now, we look at the case in which the interior region is not on the edge of the box, but within all the faces of the box surrounding the knot. However, we find that for Scenario 1 and 3 we have an outer edge that can always compose with any of the line scenarios A-F. We just take the step on the outmost corner of the interior region so that enough blank space surrounds it. That way, we do not have to worry about adding in two extra sticks to bridge the two knots. For Scenario 2, we can perform a square move as before to arrive at Scenario 1, which is given. For Scenario 4, we can also find an outer edge to compose with.

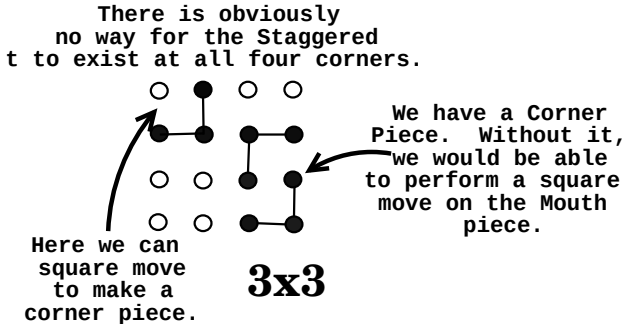
On the other hand, Scenario 5 has been found to be the only problem case. We call the structure of the corner pieces found in corner of the face, the *Downfall Situation* and the shape that the two corner pieces make, the *t shape*.

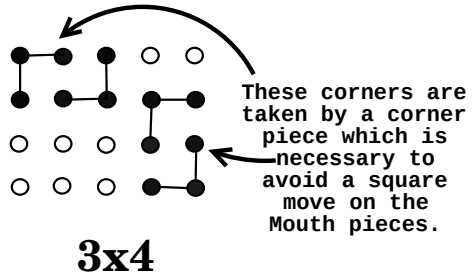


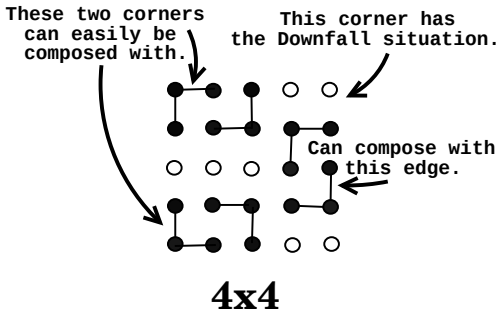
The Downfall Situation

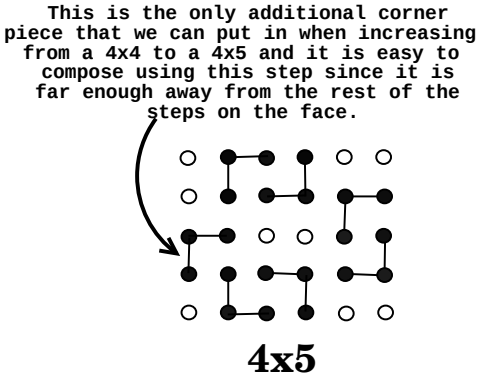
Figure 4.2.10

It can compose with any of the line scenarios except for Scenario E because the inner lines overlap with other parts of the knot. So, if we find the smallest size face, $x \times y$, for which Scenario 5 can exist, we will know that knots that fit in boxes with smaller than a $x \times y$ face can compose, even under Case 4. We found that a $9 \times 9 \times 9$ box is the smallest that will have this situation. Below, we show several cases for which Scenario E cannot occur. We start with a 3×3 face since the trefoil knot is enclosed in a $3 \times 3 \times 3$ box.









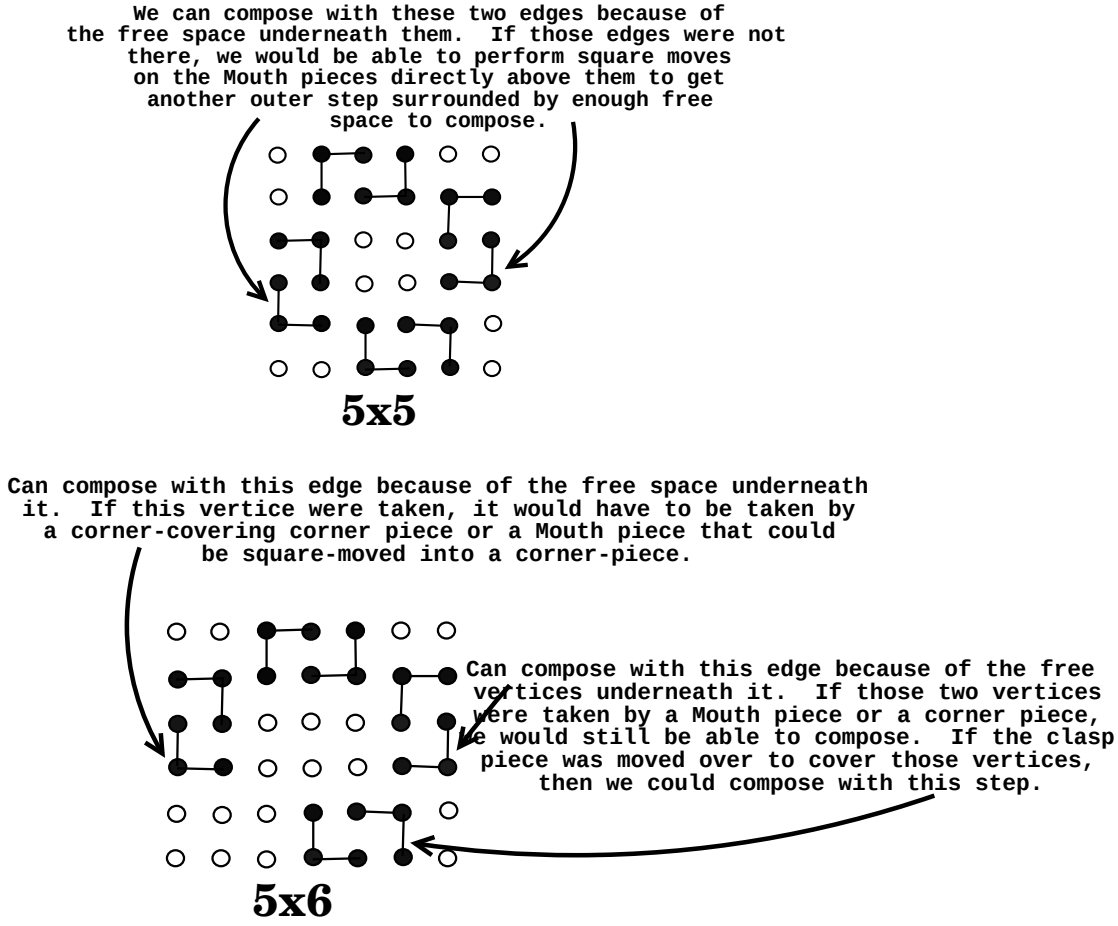
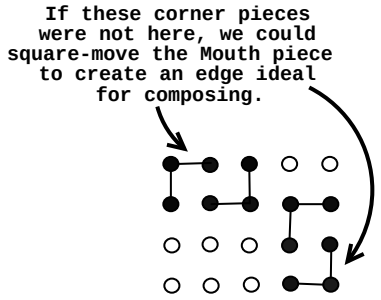


Figure 4.2.11

It is pretty obvious that any case that is $3 \times x$ will not have the downfall situation since you need at least an edge length of four steps for it to even exist on the face. Also, any case that is $4 \times x$ will not have the downfall situation on both corners of the edge of stick length 4, so we need not worry about those cases.

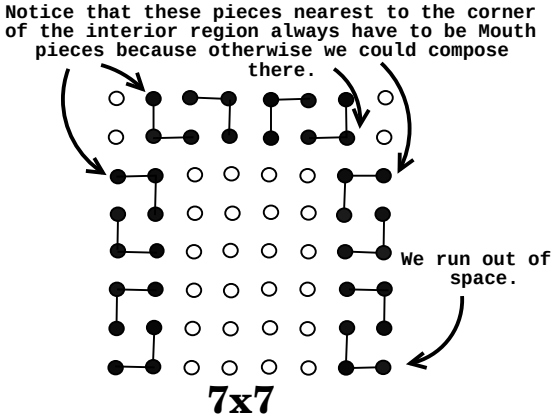
Now, from these situations in Figure , we can see general patterns. In order to have the Downfall Situation we have to have two clasps at each corner of the interior region. The reason these have to be clasps is because otherwise the mouth pieces could be square-moved to corner-pieces that could easily be composed with.



We must always have two clasps at each corner

Figure 4.2.12

Thus, we definately need each side of the interior region to be at least seven steps in order to fit in two clasps at each corner (we need a space between each of them). But, when we try to make all the sides of the interior region seven steps, we are not able to fit two clasps at each corner, as shown below.



A 7x7 is too small to have the Downfall Situation at all four corners

Figure 4.2.13

By increasing the size to a 7×8 we still cannot fit all clasps in. Yet, when we finally look at the 8×8 case we see that we can fit in two clasps at each corner in the appropriate manner.

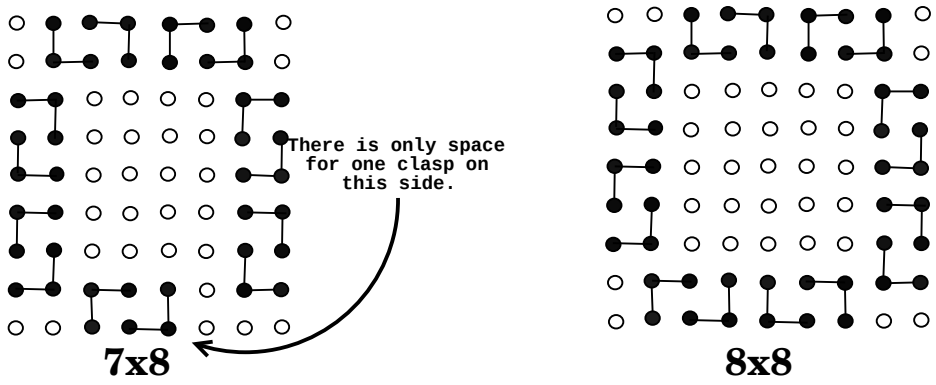


Figure 4.2.14

Thus, an interior region of at least 8 steps by 8 steps cannot compose with a cube with faces of all Scenario E. But, as we found earlier, for Scenario 4, the interior region cannot equal the face of the cube. Thus, we need a cube of at size at least $10 \times 10 \times 10$.

Therefore, for any knots in the cubic lattice surrounded by boxes smaller than $10 \times 10 \times 10$, we can conclude that, for any knot K_1 composing with a knot K_2,

$e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$. $\square$

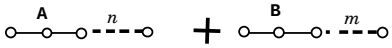
Thus, obviously, reducing the stick number by two after composition is possible as long as one knot is not a bad knot while the other has only lines such as in Scenario E. Therefore, we can say that
$e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2$ whenever both K_1 or K_2 are Good. $\square$

So, an open question remains to whether this conclusion holds true for knots surrounded by boxes of dimension greater than or equal to a $10 \times 10 \times 10$. One way to do this would be to prove that either Scenario 5 or Scenario E cannot even exist. Then, we could say that for any knot in the cubic lattice, we can lose two sticks from the total stick number every time we compose.

5 Finding a Better Upper Bound on Specific Compositions

5.1 Theorem 5.1

THEOREM 5.1: $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) + 2 - 2m$ in the situation shown below, where each knot has a line on the outer face:



Theorem 5.1 Illustrated

Figure 5.1.1

Where $m \leq n$, and n is the stick number of line A and m is the stick number of line B.

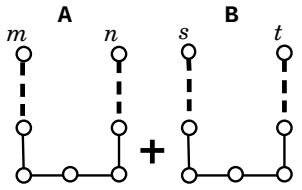
First of all, we know that $n > 1$ and $m > 1$ as stated earlier because otherwise we would be able to do a square move and reduce the stick number. So, the most basic case would be where $n = 2$ and $m = 2$ which we prove earlier to have a loss of two sticks after composition.

The first part of the theorem is known from [2], that the stick number of the composed knots will be at least the stick numbers of the two individual knots added together. Then, since a "line" shape can rotate in any way to always overlap with another "line" shape, then we can definitely remove any sticks that overlap. Since m is less than n, then we know that at least m sticks from each knot will overlap. Thus, we can take away $2m$ from the composed knot (amounting to the $-2m$ in Theorem 5.1). Then, as usual, we must always add 2 more sticks to bridge the two knots together.

Therefore, we have that $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) + 2 - 2m$. $\square$

5.2 Theorem 5.2

THEOREM 5.2: $e_\perp(K_1 \# K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2 - 2m - 2n$ in the situation shown below, where each knot has a "bowl-shaped" piece on a face of the surrounding cube, with "bottom" of the bowl always with stick number equal to 2.



Theorem 5.2 Illustrated

Figure 5.2.1

Where $m \leq s$ and $n \leq t$, m the stick number of one edge of the bowl on knot A, n the stick number of the other edge of the bowl on knot A and s and t similar for knot B. Also, the knots are oriented so the m and s overlap and n and t overlap.

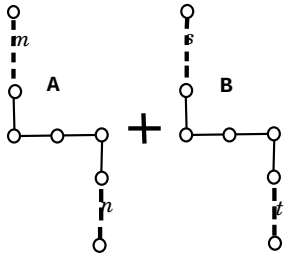
First, the two "bowls" will always be able to rotate to align the bottoms with one another. What is not guaranteed, however, is that the longest "leg" from each bowl will line up with one another to get optimal stick loss. But, this is possible. So, we overlap the two bowls to the best of our ability and remove those overlapped sticks. The least number taken out will be eight, since we have four from the bottoms of the two bowls and at least another four from the legs of the bowls. Thus, the stick number of the composed knot will start out as the stick numbers of each of the initial knots. Then, we subtract four sticks since the bottoms will always overlap, and, thus, we can always remove those sticks. Afterwards, we can remove $2m$ sticks from one side of the overlapped bowls and $2n$ from the other side of the overlapped bowls. Then, we add in 2 sticks to bridge the two knots.

Therefore, we find that $e_\perp(K_1\#K_2) \leq e_\perp(K_1) + e_\perp(K_2) - 2 - 2m - 2n$.

***Note: this method will work for any two bowl-shapes with the same bottom length. Thus, the "-4" in the equation decrease by two for each stick added to the length of the bottom piece. □

5.3 Theorem 5.3

THEOREM 5.3: $e_\perp(K_1\#K_2) \leq e_\perp(K_1)+e_\perp(K_2)-2-2m-2n$ in the situation shown below, where on the face of each knot there exists a base "trunk" of stick number two with two branches stemming off in opposite direction.



Theorem 5.3 Illustrated

Figure 5.3.1

Where $m \leq s$ and $n \leq t$, m the stick number of one branch of the figure on knot A, n the stick number of the other branch of the bowl on knot A and s and t similar for knot B. Also, the knots are oriented so that the m and s overlap and the n and t overlap.

First of all, the two figures may not always overlap for ideal stick loss. However, the trunks will always overlap so that four sticks can be removed. In this case, we are looking at when the branches from the figures overlap. So, we automatically subtract four sticks from the sum of the total stick numbers of

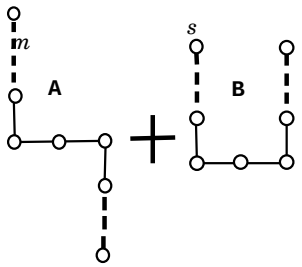
the two knots added together. Then, since $m \leq s$, we will be able to remove $2m$ more sticks. Similarly, since $n \leq t$, we can remove $2n$ more sticks (since those are the one that will overlap). Finally, add in two sticks to bridge the two knots.

Therefore, $e_{\perp}(K_1 \# K_2) \leq e_{\perp}(K_1) + e_{\perp}(K_2) - 2 - 2m - 2n$.

***Note: this method will work for any two of these figures with the same trunk length. Thus, the "-4" in the equation decrease by two for each stick added to the length of the trunk piece. □

5.4 Theorem 5.4

THEOREM 5.4: $e_{\perp}(K_1 \# K_2) \leq e_{\perp}(K_1) + e_{\perp}(K_2) - 2 - 2m$ in the situation shown below, where on the face of one knot there exists a base "trunk" of stick number two with two branches stemming off in opposite direction and on the face of the other knots there exists a "bowl-shaped" figure with the bottom of the bowl of length two and two branches stemming off in opposite directions.



Theorem 5.4 Illustrated

Figure 5.4.1

Where $m \leq s$, m the stick number of one branch of the figure on knot A and s the stick number of one branch on the figure on knot B. Also, assume the knots are oriented so that the m and s overlap.

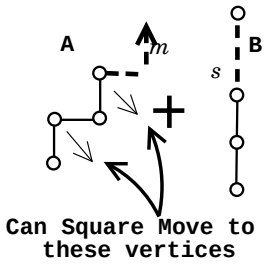
First, since the bowl-shaped figure of knot A can rotate to always align with part of the figure on knot B, we will always be able to remove at least six sticks: the trunk pieces, and two of the overlapping branches. So, from the initial stick numbers of the two knots added together, after composition, we can subtract four sticks from the trunk pieces. Then, since $m \leq s$ and those two pieces overlap, we can remove $2m$ more pieces. And, finally, we add two sticks to bridge the two knots together.

Therefore, we get $e_{\perp}(K_1 \# K_2) \leq e_{\perp}(K_1) + e_{\perp}(K_2) - 2 - 2m$.

***Note: this method will work for any two of these figures with the same trunk length. Thus, the "-4" in the equation decrease by two for each stick added to the length of the trunk piece. □

5.5 Theorem 5.5

THEOREM 5.5: $e_\perp(K_1\#K_2) \leq e_\perp(K_1) + e_\perp(K_2) + 2 - m$ when m is even and $e_\perp(K_1\#K_2) \leq e_\perp(K_1) + e_\perp(K_2) - m$ when m is odd in the situation shown below, where on the face of one knot there exists a staircase surrounded by open vertices and on the face of the other knot there exists a straight line.



Theorem 5.5 Illustrated

Figure 5.5.1

Where m is the stick number of the figure in knot A and s is the stick number of the straight line. $m < s$.

So, we start with the sum of the initial stick numbers of the two knots. Then, since the corner piece is surrounded by open vertices, we can perform square moves until we end up with two straight lines perpendicular to one another. Then, if m is even, we will be able to take $m/2$ sticks and overlap those (because they are in a straight line) with the straight line on knot B. Then, we can remove twice $m/2$ sticks from the original stick numbers of the two knots. Thus, we remove m sticks. Finally, we add in two more sticks to bridge the two knots.

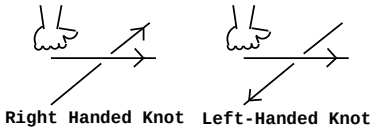
If, on the other hand, m is odd, we instead overlap $m/2 + 1$ sticks. Thus, we remove $m + 2$ sticks and then add in two more to bridge the two knots.

Therefore, we have that $e_\perp(K_1\#K_2) \leq e_\perp(K_1) + e_\perp(K_2) + 2 - m$ when m is even and $e_\perp(K_1\#K_2) \leq e_\perp(K_1) + e_\perp(K_2) - m$ when m is odd. □

6 Composing Trefoil Knots on the Cubic Lattice

6.1 Definitions

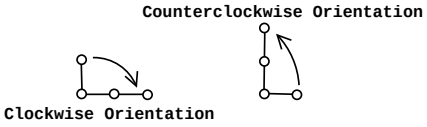
Definition 1: Let a knot have a certain fixed direction that it travels along itself. Then, use your right hand and, at a crossing, point your thumb in the direction that the top strand goes. If your fingers curl in the direction that the lower strand points, then the crossing has $right - handed\ orientation$. If your fingers curl in the opposite direction to that of the bottom strand, then the crossing is a $left - handed$ crossing[1].



Definition 1 Illustrated

Figure 6.1.1

Definition 2: Given a knot, look down one axis of the knot. We look for the L-shaped pieces on the front face of the knot. If we go clockwise when traveling from short piece to long piece, then that piece of the knot has *Clockwise orientation*. Similarly, a piece of the knot has *Counterclockwise orientation* if one travels counterclockwise from short to long piece.



Definition 2 Illustrated

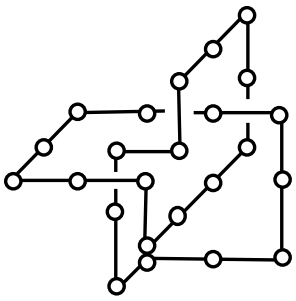
Figure 6.1.2

6.2 Theorem 6.1

THEOREM 6.1: $e_{\perp} \overset{\#}{n} (3\ \overset{+}{1}\ \#3\ \overset{-}{1}) \leq 24 + (n-1)16$ where n is the number of trefoil knots in the cubic lattice.

PROOF:

We know that the trefoil knot has 24 sticks (shown below)[3].

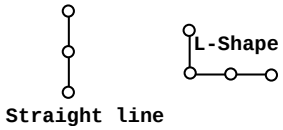


The Trefoil Knot

Figure 6.2.1

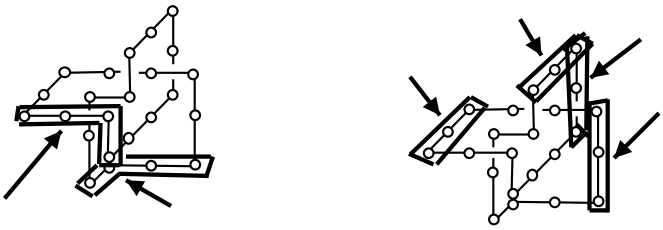
Now, put a box around the trefoil and notice the shapes that appear on each face. Basically there are two shapes: a line of length two and an L-shape. Thus,

we have three ways to compose: an L-shape with an L-shape, a line with a line, and a line with an L-shape.



The Shapes on the Faces of the Trefoil Knot

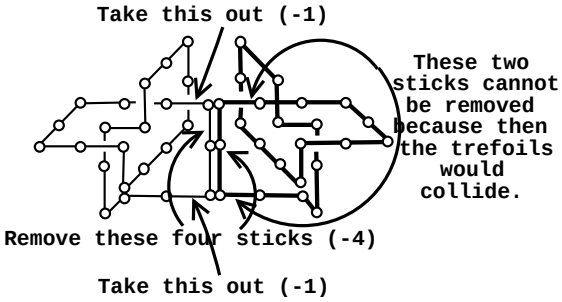
Figure 6.2.2



The L-shapes The Sticks Shapes

Figure 6.2.3

If we try to compose a line with a line, we find that we can only lose six sticks in the process because of the tight structure of the trefoil (shown below). A similar situation arises with composing the line with the L-shape. In these two situations, we overlap the two lines of two, removing four sticks, and then we can get rid of two of the bridging sticks so that the two knots are as close together without overlapping.



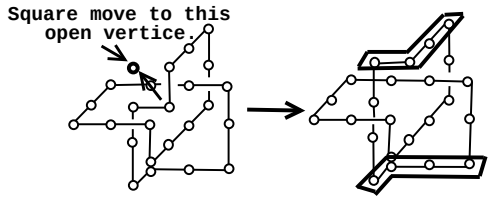
Cannot Compose Along Two Straight Lines

Figure 6.2.4

However, when we compose an L-shape with another L-shape, six sticks overlap, and then we can take away two more without the two trefoils colliding. But, in order for two L's to overlap, they must be of opposite orientation, that is,

one must be clockwise, the other counterclockwise. Note that the right-handed trefoil has counterclockwise orientation and the left-handed trefoil has clockwise orientation (See Figure 6.2.6). So, obviously, it seems natural to be able to compose a line of knots such that they are alternating, right-handed and left-handed.

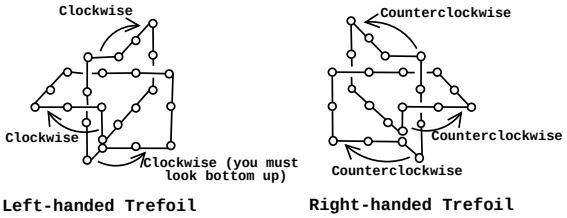
Now, a small problem arises. In the drawing of the trefoil given in [3], we have two L-shapes that are on adjacent sides. If we tried to compose a chain, the knots would eventually run into one another. Thus, we perform a square move, as shown below, to get two L-shapes directly across from one another.



Square Move to get Two L-shapes Directly Across from One Another

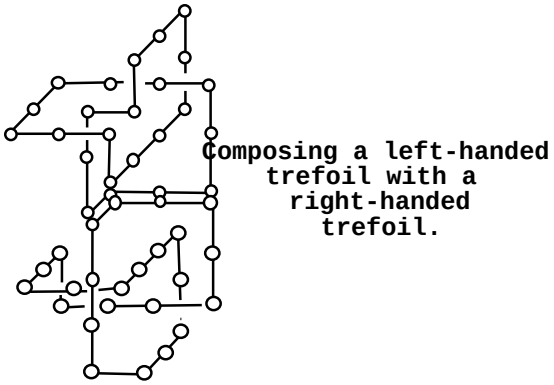
Figure 6.2.5

Since we are alternating knots, and since the right-handed knot has a minimal stick representation with counterclockwise orientation and the left-handed knot has only clockwise orientation (shown below), then we can easily compose them in a straight line. All we need is for a clockwise L to line up with a counterclockwise L which can obviously be done in this manner.



The Orientation is Consistent Throughout A Single Trefoil

Figure 6.2.6



Composing two trefoils

Figure 6.2.7

So, obviously, we lose eight sticks when two trefoils compose, or, we must add 16 sticks for each additional trefoil. Since the initial stick number is 24, we add 16 sticks for each additional trefoil. Thus, if there are n trefoils, we take $16(n-1)$ to account for all but the initial. Then, we add 24 to account for the initial. Therefore, for any chain of trefoils composed, we have that:

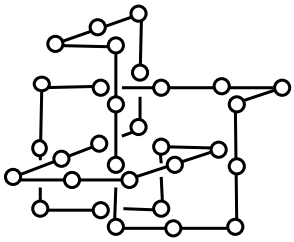
$e_{\perp}(\overset{\#}{n}\, 3_1) \leq 24 + (n-1)16.$

Therefore, an open question remains as to whether or not we can get the same results with only a right-handed trefoil. Thus far, no one has conjectured to the stick number of the composition of two right-handed trefoils, so this is also yet to be found. □

7 Composition of Figure-Eight Knots on the Cubic Lattice

7.1 Definitions

Definition 1: The $figure-eight$ knot is one which has four crossings. An upper bound for the stick number of the figure-eight in the cubic lattice is 30 sticks[4].



The Figure-Eight Knot in the Cubic Lattice

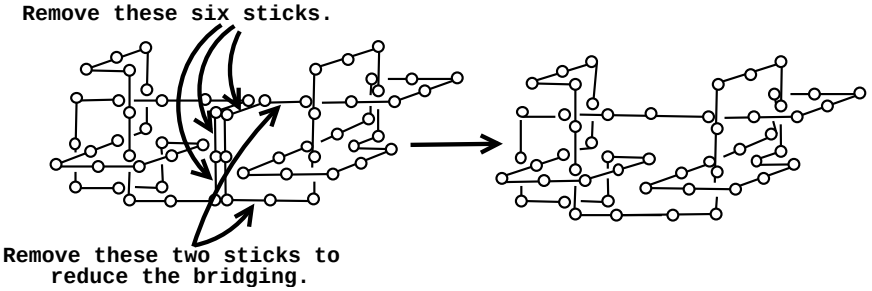
Figure 7.1.1

7.2 Theorem 7.1

THEOREM 7.1: $e_{\perp}(K_1 \# K_2) \leq e_{\perp}(K_1) + e_{\perp}(K_2) - 8$ for two figure-eight knots.

PROOF:

We can use a similar method of proving this that we used to prove Theorem 4.1. Notice that the figure-eight has two L-shapes on adjacent sides. Luckily, these Ls are of different orientation, one is counterclockwise and the other is clockwise. Thus, we can compose two knots that are either both right-handed, both left-handed or a combination of the two. So, when we overlap the two Ls, we automatically lose the six sticks from the actual L-shape and then we can lose two more of the bridging sticks without worrying about the knots colliding.



Composing Two Figure Eights

Figure 7.2.1

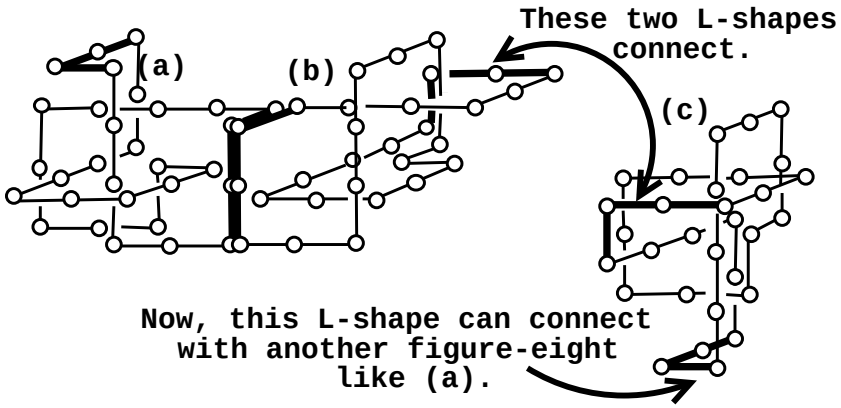
Therefore, it becomes obvious that:

$e_{\perp}(K_1 \# K_2) \leq e_{\perp}(K_1) + e_{\perp}(K_2) - 8$ for two figure-eight knots. □

7.3 Theorem 7.2

THEOREM 7.2: $e_{\perp}(\overset{\#}{n}\, 4_1) \leq 30 + (n-1)22$ where n is the number of figure-eight knots.

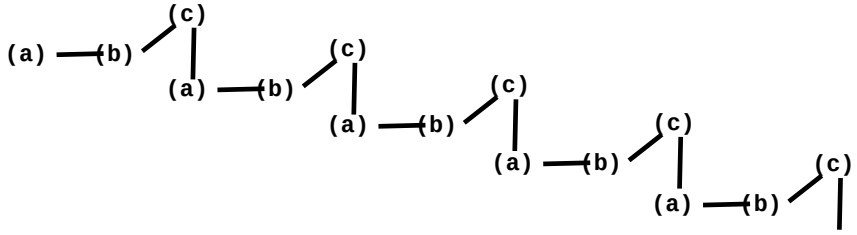
So, we know that the upper bound on stick number for the figure-eight is 30 sticks. We use the same technique used in proving Theorem 6.1. First, we draw a box around the figure-eight. Then, we easily see that there are two types of shapes on the faces of the box: sticks of length two and L-shapes. We already know that composing sticks of length two yields only a loss of six sticks, so we can try to compose the L-shapes. However, unlike in Theorem 6.1, we cannot use square moves to make two Ls across from one another. Nevertheless, when we start composing the-figure eights in a chain, we find that after three different orientations of the knot in 3-space, a pattern forms and repeats itself, as illustrated below.



Composing Several Figure Eights

Figure 7.3.1

And, as the picture illustrates, (a) composes with (b) which composes with (c) which in turn composes with (a), starting the chain over again. Thus we have a pattern like so:



Pattern Created when Figure-Eight Knots Compose in a Chain

Figure 7.3.2

Now, we know that eight sticks are lost between two figure-eight knots. Thus, 30 Sticks - 8 sticks yields 22 sticks left for each knot except for the first and the last ones (since those two only compose once). Then, we have that the total stick number is $22n + 8$. We add in the eight since the first and the last knots each have 26 sticks rather than 22. Then, we can rewrite $+8$ as $-22 + 30$ and then the math follows:

$22n + 8 \Rightarrow$

$22n(30 - 22) \Rightarrow$

$24 + 22n - 22 \Rightarrow$

$24 + 22(n - 1) \Rightarrow$

which is what is needed.

Therefore, we have that for a chain of figure-eights,

$e_{\perp}(\overset{\#}{n}\, 4_1) \leq 30 + (n - 1)22$ where n is the number of figure-eight knots. $\square$

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