Three-Variable Bracket Polynomial for Two-Bridge Knots

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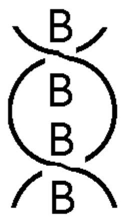
Abstract

In this paper, we derive recursive formulas for a twist connected to a tangle and for a twist that connects to a tangle in two places. We use these formulas to derive the three-variable bracket polynomial for two-bridge knots. This derivation allows us to calculate the number of states corresponding to the maximal exponent of d in a two-bridge knot with no single crossings.

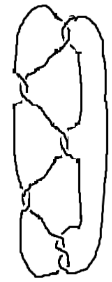
Introduction

A knot is simply defined as a closed curve in R^3. An alternating knot is a knot where the crossings alternate between going under and going over. A reduced alternating diagram of a knot can be thought of as the simplest way to depict the knot which minimizes the number of crossings. A flype is a move that creates a twist in the knot. The Tait Flyping Theorem states that two reduced alternating diagrams of the same knot can be deformed into each other by a sequence of flypes. The crossing number is defined as the number of crossings in a knot. A twist is a portion of the knot with a set of consecutive crossings. An A-twist is a twist where the A regions are contained inside the twist. An A-region is a region above a crossing where the overcrossing goes to the left. A B-twist is a twist where the B regions are contained inside the twist. A B-region is a region above a crossing where the overcrossing goes to the right. The twist number is defined as the number of twists in a knot. A two-bridge knot is a knot which has only two maxima and minima. A torus knot is a knot that can be embedded onto the surface of a torus. A two-torus knot is a knot that can be embedded onto the surface of a 2-torus. The three variable bracket polynomial of a reduced alternating knot is invariant under flypes.

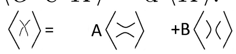
A-Twist



B-Twist



2-Bridge Knot

The three- variable bracket polynomial is one way of describing a reduced alternating diagram of a knot uniquely. The three-variable bracket polynomial of a knot can be computed by using the following rules:
1) $\langle O \rangle = 1$.
2) $\langle O \text{ U } K \rangle = \text{d } \langle K \rangle$.
3) 

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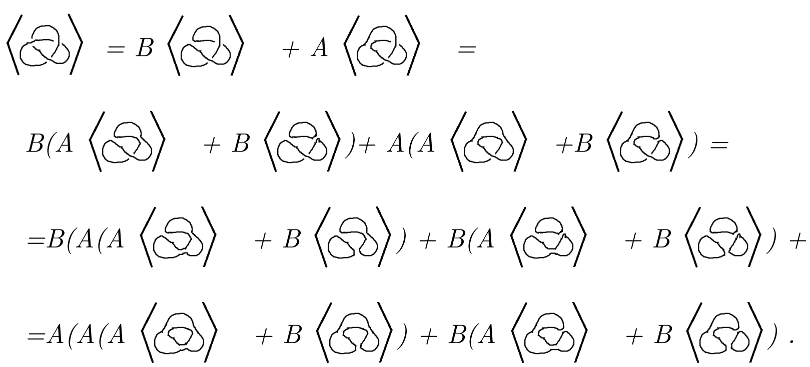
Example 1. *The three variable bracket polynomial of the trefoil in Figure 1 is found in the following way,*

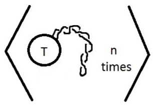


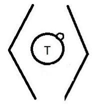
$= A^3d + 3AB^2d + 3A^2B + B^3d^2.$

Figure 1

Results

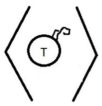
Lemma 1. *For a B-twist, the three variable bracket polynomial is given by,*

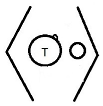


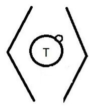
$= (Ad+B)^n$


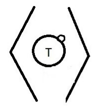
.

Proof. Let n=1, then we have that,



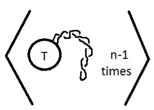
= A


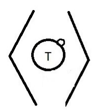
+ B


= (Ad+B)


.

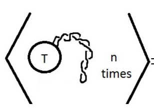
Therefore, the formula works when n=1. Now suppose that the formula is true for n-1, then we have that,

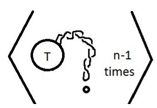


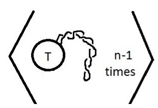
$= (Ad+B)^{n-1}$


.

We also have the following recursive relation,



= A


+ B


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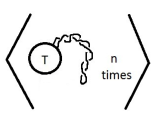
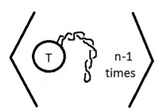


$= (Ad+B)$


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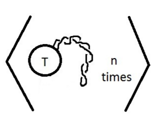
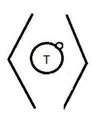
Combining the induction hypothesis with the recursive relation, we arrive at the formula,



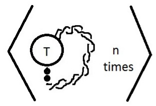
$= (Ad+B)^n$


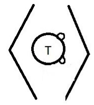
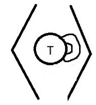
.

Therefore, by induction, the lemma must be true.

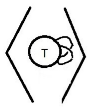
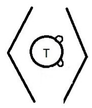
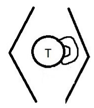
□

Lemma 2. *For a B-twist, the three-variable bracket polynomial is given by,*



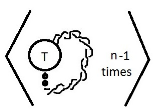
$= \dfrac{(Ad+B)^n - B^n}{d}$

$+ B^n$

.

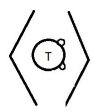
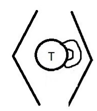
Proof. Let n=1, then the recursive definition of the three variable bracket polynomial gives,


= A

+ B

.

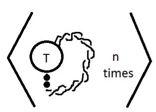
Therefore, the statement is true for n=1.

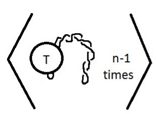
Now suppose that the statement is true for n-1, then,

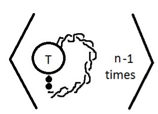


$= \dfrac{(Ad+B)^{n-1} - B^{n-1}}{d}$

$+B^{n-1}$

.

We also have the following recursive relation,



= A


+ B


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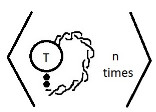
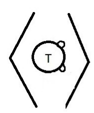
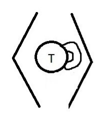
Using Lemma 1 and combining the induction hypothesis with the recursive relation above, we arrive at the following formula,

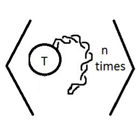


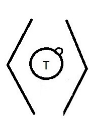
$= \dfrac{(Ad+B)^n - B^n}{d}$

$+B^n$

.

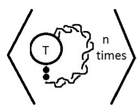
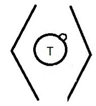
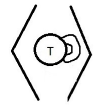
Therefore, by induction, the lemma must be true.

□

For A twists, we simply change the A's to B's and the B's to A's in the above formulas. The proof is left to the reader. The following formulas apply to A-twists.



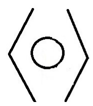
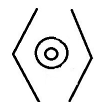
$= (Bd+A)^n$

.

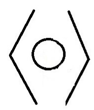
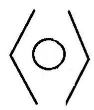
 $= \frac{(Bd+A)^n - A^n}{d}$  $+ A^n$  .

These formulas allow us to calculate the bracket polynomial for two bridge knots quickly. The example below demonstrates this.

Corollary 1. *For a two-torus knot, $T_{2,n}$, the three variable bracket polynomial is given by,*

$$\langle T_{2,n} \rangle = \frac{(Ad+B)^n + (d^2-1)B^n}{d}.$$

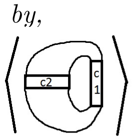
Proof. $\langle T_{2,n} \rangle = \frac{(Ad+B)^n - B^n}{d}$  $+ B^n$  .

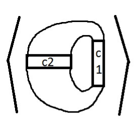
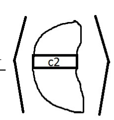
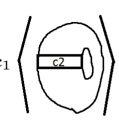
$= \frac{(Ad+B)^n - B^n}{d}$  $+ dB^n$  .

$$= \frac{(Ad+B)^n + (d^2-1)B^n}{d}$$

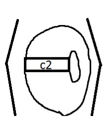
□

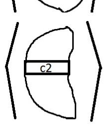
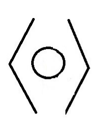
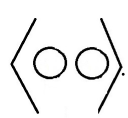
Lemma 3. *Suppose we have the knot shown in the figure below, where c_1 is an A-twist and c_2 is a B-twist, the three-variable bracket polynomial is then given by,*

 $= \frac{(Bd+A)^{c_1} - A^{c_1}}{d} \left(\frac{(Ad+B)^{c_2} - B^{c_2}}{d} + dB^{c_2} \right) + A^{c_1}((Bd+A)^{c_2}).$

Proof.  $= \frac{(Bd+A)^{c_1} - A^{c_1}}{d}$  $+ A^{c_1}$  .

Note that,

 $= (Bd+A)^{c_2}.$

 $= \frac{(Ad+B)^{c_2} - A^{c_2}}{d}$  $+ B^{c_2}$  .

Substituting the second and third equations into the first equation, we get the equality in the lemma.

$\square$

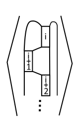
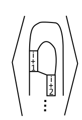
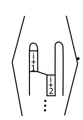
In the theorem to follow, we mean $\langle L(c_1, c_2, c_3, ...c_n)\rangle$ to be the three variable bracket of a two-bridge knot with twists. Where c_1 is the number of crossings in the first twist, c_2 is the number of crossings in the second twist, and so on.

$\alpha_{c_i} = \dfrac{(Ad+B)^{c_i} - B^{c_i}}{d}$ if c_i and $\beta_{c_i} = B^{c_i}(Bd+A)^{c_{i+1}}$ if c_i is a B-twist.
$\alpha_{c_i} = \dfrac{(Bd+A)^{c_i} - A^{c_i}}{d}$ if c_i and $\beta_{c_i} = A^{c_i}(Ad+B)^{c_{i+1}}$ if c_i is an A-twist.

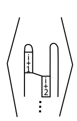
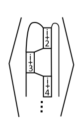
Theorem 1. *The three variable bracket polynomial of any two-bridge knot can be recursively defined as,*

$$\langle L(c_i, c_{i+1}, c_{i+2}, ...c_n)\rangle = \alpha_{c_i}\langle L(c_{i+1}, c_{i+2}, c_{i+3}, ...c_n)\rangle + \beta_{c_i}\langle L(c_{i+2}, c_{i+3}, c_{i+4}, ...c_n)\rangle.$$

Proof. We will treat the two bridge knot with an odd number of twists first. We can assume that the top twist is an A-twist. By Lemma 2, we have that,

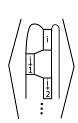
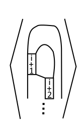
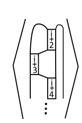
 $= \dfrac{(Bd+A)^{c_i} - A^{c_i}}{d}$  $+\, A^{c_i}$ .

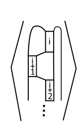
We also know that,

 $=$  $(Ad+B)^{c_{i+1}}$.

By Lemma 1 and knowing that box $i+1$ is a B-twist.

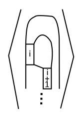
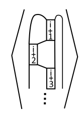
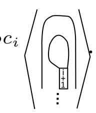
Substituting, we arrive at the following equality,

 $= \dfrac{(Bd+A)^{c_i} - A^{c_i}}{d}$  $+\, A^{c_i}(Ad+B)^{c_{i+1}}.$ .

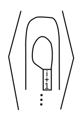
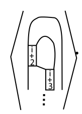
Letting $\langle L(c_i, c_{i+1}, ...c_n)\rangle =$ , we arrive at the following result,

$\langle L(c_i, ...c_n)\rangle = \alpha_{c_i}\ \langle L(c_{i+1}, ...c_n)\rangle\ +\ \beta_{c_i}\ \langle L(c_{i+2}, ...c_n)\rangle.$

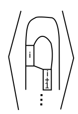
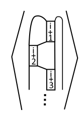
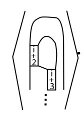
Next, we will look at the case when the two bridge knot has an even number of twists. We know that in this two bridge knot, the top twist must be a B-twist. By Lemma 2, we have that,

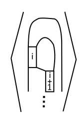
 $= \dfrac{(Ad+B)^{c_i} - B^{c_i}}{d}$  $+\ B^{c_i}$ 

By Lemma 2 and knowing that box $i+2$ is an A-twist,

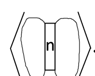
 $= (Bd+A)^{c_{i+1}}$ 

Substituting, we arrive at the following result,

 $= \dfrac{(Ad+B)^{c_i} - B^{c_i}}{d}$  $+\ B^{c_i}\ (Bd+A)^{c_{i+1}}$ 

Letting $\langle L(c_i, c_{i+1}, ...c_n)\rangle =$ , we arrive at the following result,

$\langle L(c_i, ...c_n)\rangle = \alpha_{c_i}\ \langle L(c_{i+1}, ...c_n)\rangle\ +\ \beta_{c_i}\ \langle L(c_{i+2}, ...c_n)\rangle.$

We notice that $\langle L(c_n)\rangle =$ . This is just a 2-torus knot where c_n is an A-twist. Therefore, the polynomial $\langle L(c_n)\rangle = \dfrac{(Bd+A)^{c_n} + (d^2-1)A^{c_n}}{d}$.

□

Theorem 2. *The initial conditions for the theorem above are that with* $\langle L(c_n)\rangle = \dfrac{(Bd+A)^{c_n} + (d^2-1)A^{c_n}}{d}$ *and* $\langle L(c_2)\rangle = \frac{(Bd+A)^{c_1} - A^{c_1}}{d}\ (\frac{(Ad+B)^{c_2} - B^{c_2}}{d}\ + dB^{c_2}) + A^{c_1}((Bd+A)^{c_2})$.

Proof. We can untwist the knot until we get the bottom twist, and the bottom twist is just a torus knot of type (2,n). This torus knot has the equation from the lemma which happens to be the equation stated above. Now the second initial condition is derived from the fact that we have two twists in a torus knot. One happens to be an A-twist, the other happens to be a B-twist and we arrive at the formula above. This follows directly from the lemma.

□

From the previous theorem, we see that the three variable bracket polynomial is a summation of a bunch of sequences of α's and β's. For instance, $\alpha_{c_1}\alpha_{c_2}...\alpha_{c_n}$ is a term in the three variable bracket polynomial. We also notice that we can obtain other terms by replacing any two consecutive α's with one β. For instance, $\alpha_{c_1}\beta_{c_2}\alpha_{c_4}...\alpha_{c_n}$ and $\alpha_{c_1}\beta_{c_2}\beta_{c_4}\alpha_{c_6}...\alpha_{c_n}$ are two such terms in the polynomial.

Example 2. *The three-variable bracket polynomial of the two-bridge knot $L(3,4,4)$ is equal to $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3} + \beta_{c_1}\alpha_{c_3} + \alpha_{c_1}\beta_{c_2}$ where $\alpha_{c_1} = \frac{(Bd+A)^3+(d^2-1)A^3}{d}$, $\alpha_{c_2} = \frac{(Ad+B)^4-B^4}{d}$, $\alpha_{c_3} = \frac{(Bd+A)^4-A^4}{d}$, $\beta_{c_1} = A^3(Ad+B)^3$, and $\beta_{c_2} = B^4(Bd+A)^4$.*

Theorem 3. *Consider a two bridge knot with all twists having more than one crossing each. The maximal degree of d in the bracket polynomial of the two-bridge knot is equal to the crossing number minus the twist number.*

Proof. Consider the all alpha term, $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3}.....\alpha_{c_n}$. The highest exponent of d here is $C-T$. This is obtained by adding up all the terms, $c_1, c_2, ...c_n$ and subtracting the twist number. Now suppose we replace α_{c_k} with β_{c_k}, then we delete $\alpha_{c_{k+1}}$ as well. Since all twists have two or more crossings, the exponent of d in the resulting sequence has to be less than or equal to $C-T$. Therefore any sequence in the three variable bracket polynomial must have the highest exponent of d less than or equal to $C-T$. We can conclude from this that the maximal degree of the bracket polynomial is equal to the crossing number minus the twist number in this case.

□

In the next theorems, we mean a block of 2-twists to be a set of consecutive twists in a two-bridge knot with 2 crossings each. We mean the length of a block of 2-twists to be the number of 2-twists in the block. For example, in Figure 2, the two-bridge knot has a block of 2-twists with length 4.

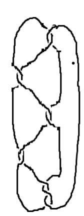


Figure 2

Theorem 4. *Let L be a two-bridge knot with no twist of single crossings and with the bottom twist having three or more crossings. The number of states corresponding to the maximal degree of d is denoted by, $F_{n_1+1}F_{n_2+1}......F_{n_k+1}$. Where n_1 is the length of the first block of 2-twists, n_2 is the length of the second block of 2-twists, and so on.*

Proof. Consider the two bridge knot $L(c_1, c_2, ...c_n)$, we claim that $F_{n_1+1}...F_{n_k+1}$ is equal to the number of sequences in the three variable bracket polynomial having a highest power of d of $C-T$. We will first prove this by induction by considering a two-bridge knot with one block of 2-twists, and increasing this block by a length of 1. Suppose we have a two-bridge knot with only one 2-twist and without twists of single crossings. We know that one such term in the three-variable bracket polynomial is $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3}.....\alpha_{c_n}$. This clearly has a highest exponent of d of $C - T$. Now consider the term $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3}.....\alpha_{c_{k-1}}\beta_{c_k}\alpha_{c_{k+2}}.....\alpha_{c_n}$ where $\alpha_{c_{k+1}}$ is a 2-twist. Calculation shows that this term also has an exponent of d of $C - T$. If we replace any other α with a β, the exponent will go down. Hence there are only two possible terms corresponding to the highest exponent of d of $C - T$. This is indeed equal to $F_{1+1} = F_2$. Now suppose we have a two-bridge knot with only one block of two-twists of length n, and suppose that the number of terms corresponding to the highest exponent of d is F_{n+1}. Block $= c_j, c_{j+1}......c_{j+n-1}$. Now suppose we increase the length of the block by adding on one more 2-twist. Then we add on c_{j+n}. Now consider the term, $\alpha_{c_1}\alpha_{c_2}......\alpha_{c_j}\alpha_{c_{j+1}}......\alpha_{c_{j+n}}....\alpha_{c_l}$. If we replace $\alpha_{c_{j+n-1}}$ with $\beta_{c_{j+n-1}}$, then we delete $\alpha_{c_{j+n}}$, and the term becomes $\alpha_{c_1}\alpha_{c_2}......\alpha_{c_j}\alpha_{c_{j+1}}......\beta_{c_{j+n-1}}\alpha_{c_{j+n+1}}....\alpha_{c_l}$. Now we are dealing with the block $= c_j, c_{j+1}......c_{j+n-2}$. And the number of terms corresponding to the highest power of d here is F_n. In addition, we can have the term, $\alpha_{c_1}\alpha_{c_2}......\alpha_{c_j}\alpha_{c_{j+1}}......\alpha_{c_{j+n}}\alpha_{c_{j+n+1}}....\alpha_{c_l}$. And the number of terms corresponding to the highest exponent of d here is F_{n+1}. In total, the number of terms corresponding to the highest exponent of d in the resulting block is $F_n + F_{n+1} = F_{n+2}$. Which is precisely the Fibonacci sequence. Now suppose we have a two-bridge knot with only one block of 2-twists of length n. It was shown that the amount of states corresponding to the maximal degree is F_{n+1}. Now suppose we have k blocks of two twists of length $n_1, n_2, n_3....n_k$ respectively. And suppose that the number of terms corresponding to the maximal degree of d is $F_{n_1+1}F_{n_2+1}F_{n_3+1}......F_{n_k+1}$. Now suppose we add on another block so we have $k + 1$ blocks in total. For each sequence on the $k + 1$ block we have $F_{n_1+1}F_{n_2+1}F_{n_3+1}......F_{n_k+1}$ sequences, so in total we have $F_{n_1+1}F_{n_2+1}F_{n_3+1}......F_{n_{k+1}+1}$ sequences corresponding to the $k+1$ blocks of 2-twists. In conclusion, the theorem is true.

□

Theorem 5. *Let L be a two bridge knot with no twist having a single crossing or a double crossing and with the bottom twist having three or more crossings. The number of states corresponding the maximum degree of d minus one is equal to $n + c$, where n is the number of 3-twists.*

Proof. Consider the two-bridge knot, $L(c_1, c_2, ...c_n)$. Now consider the all alpha state, $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3}.....\alpha_{c_n}$. The exponent of d here is $C-T$. And the second highest exponent of d is $C-T-1$. Now suppose we replace α_k with a β_k where α_{k+1} is a 3-twist. Then our new term is $\alpha_{c_1}\alpha_{c_2}\alpha_{c_3}.....\beta_{c_k}\alpha_{c_{k+2}}..\alpha_{c_n}$. Since α_{k+1} is not in the term, the exponent of d goes down by three. But the exponent of d also goes up by one since β_k replaces α_k. Therefore the exponent of d in the above term is $C - (T - 1) - 3 + 1 = C - T - 1$. Therefore, this state contributes 1. If we replace any other α with a β, then the resulting exponent of d must be less than $C - T - 1$. So we can only replace two alphas with one beta. But we can do this with all 3-twists, so the number of states contributed to the second highest exponent of d must be equal to n where n is the number of 3-twists. We can also look at the all alpha term, clearly the number of states corresponding to the second highest exponent of d is equal to c. So in total, there are $n + c$ states corresponding to the second highest power of d.

□

Open Questions

How can we extend these results to two-bridge knots with single crossings? That is, what are the number of states corresponding to the highest exponent of d and the second highest exponent of d in a two-bridge knot if single crossings are allowed?

What form do the states take on corresponding to the highest exponent of d and the second highest exponent of d?

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