MATH 329-01: Transformation Geometry (82868)
JB-387, TuTh 6 - 7:50 PM
SYLLABUS Fall 2017

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Office Hours: TuTh 3:30 - 5:30 PM, or by appt.

Recommended References:
Brannan/Esplen/Gray (Cambridge)
Geometry (second edition)
ISBN 978-1-107-64783-1

Coxeter/Greitzer, Geometry Revisited MAA New Mathematical Library (ISBN: 0883856190)

The Thirteen Books of Euclid’s Elements (3 vol., Dover).

Prerequisites: MATH 251, and high-school geometry or equivalent

This is an upper-division course in Euclidean geometry that serves two purposes: 1) develop the language of geometric transformations, classify those which establish congruence in the Euclidean plane, and apply them to some familiar theorems; 2) explore some results that are not typically encountered in a first geometry course. We will approach these goals concurrently through lectures, reading, and your participation in presentations. The required text listed above is primarily for reference. There is useful background in the first two chapters and the remaining chapters develop material that is used in future courses such as MATH 529 and MATH 614. However, I will structure this course from my own notes in a way that fits the above two purposes. Those notes will be posted on my website (math.csusb.edu/faculty/sarli) as the course progresses. Occasionally I may suggest exercises and reading from Chapter 2 of the text.

Important. You will find very few diagrams in my notes. I encourage you to draw your own diagrams as you read through the development. Use a compass and straightedge for accuracy. There are fifty Exercises in the body of these notes. You should work through many of them as they arise and keep notes of your solutions for reference. In addition, there are three junctures where Suggested Exercises for Study are given. Use these to create practice exams for yourself.
Grading will be based on two exams (midterm and final), and two graded assignments. The midterm and final exams will be based on the material we develop for geometric transformations. Your course grade will be determined as follows:

1) First graded assignment (10%)
2) Midterm exam (30%)
3) Second graded assignment (20%)
4) Final exam (40%)

For the two in-class exams you should bring your notes so that you learn how to use your own work when processing the material.

After computing your total scores weighted according to these percentages, course grades will be assigned as follows:

\[
\begin{align*}
A & \geq 91 \\
A- & 86 - 90 \\
B+ & 81 - 85 \\
B & 76 - 80 \\
B- & 71 - 75 \\
C+ & 66 - 70 \\
C & 61 - 65 \\
C- & 51 - 60 \\
D & 45 - 50 \\
F & < 45 
\end{align*}
\]
Notes

1) Learning Outcomes: Upon successful completion of this course, students will be able to:
   1.2 make connections between mathematical ideas verbally, numerically, analytically, visually, and graphically;
   2.1 correctly apply mathematical theorems, properties and definitions;
   3.6 critique mathematical reasoning, both correct and flawed;
   5.1 understand correct mathematical proofs.

2) The mid-term exam date is subject to change.

3) Please refer to the Academic Regulations and Policies section of your current bulletin for information regarding add/drop procedures. Instances of academic dishonesty will not be tolerated. Cheating on exams or plagiarism (presenting the work of another as your own, or the use of another person’s ideas without giving proper credit) will result in a failing grade and sanctions by the University. For this class, all assignments are to be completed by the individual student unless otherwise specified.

4) If you are in need of an accommodation for a disability in order to participate in this class, please let me know ASAP and also contact Services to Students with Disabilities at UH-183, (909)537-5238.

Some important dates:

09-21: First day of class
10-11: Census Deadline
10-17: Final draft of first graded assignment due
10-19: Midterm Exam
11-23: Campus Closed
11-30: Last day of class; Final draft of second graded assignment due
12-07: Final Exam (Thursday 6-7:50)
Here is the approximate schedule for this ten-week course:

Week 1
Some History

Week 2
Why We Need Transformations

Week 3
Translations and Rotations

Week 4
Isometries of the Euclidean Plane, I

Week 5
Isometries of the Euclidean Plane, II

Week 6
Composition of Finitely Many Reflections

Week 7
Classification of Rigid Motions

Week 8
Similarity Transformations

Week 9
Fundamental Theorem of Affine Geometry

Week 10
Transformation of General Figures
Notation

Ordinary upper-case letters \((P, Q, \ldots)\) refer to points in the plane.

A figure is any collection of points. A figure consisting of more than one point is denoted by an upper-case script letter \((\mathcal{F}, \mathcal{C}, \ldots)\), except that we use \(l, m, \ldots\) for lines and we may write \(l = \overrightarrow{AB}\) for the line through the distinct points \(A, B\). Specific rectilinear figures are usually written in terms of points that define them:

\[
\overrightarrow{AB} \text{ is the line through the distinct points } A, B \text{ (so we may write } l = \overrightarrow{AB}).
\]

\[
\overline{AB} \text{ is the segment between } A, B.
\]

\[
\overrightarrow{AB} \text{ is the ray with endpoint } A.
\]

The angle \(\angle ABC\) is the union of rays \(\overrightarrow{BA}\) and \(\overrightarrow{BC}\).

The triangle with vertices \(A, B, C\) is written \(ABC\), but see the note, below, on orientation.

The length of segment \(\overline{AB}\) is \(AB\), a non-negative quantity. When context requires oriented measurement of a segment the length is replaced by distance \(AB\) from \(A\) to \(B\), whereby \(BA = -AB\).

The measure of \(\angle ABC\) is denoted by \(\angle ABC\), taken to be in \([0, \pi]\) unless stated otherwise. When context requires oriented measurement of an angle we have \(\angle CBA = -\angle ABC\).

A transformation is a 1-1 and onto mapping of the points of the plane to themselves. The generic notation for a transformation is \(T\), but it will be convenient to refer to the reflection in line \(l\) by the letter \(l\) itself (context will make the distinction between figure and mapping).

A point \(P\) is fixed by a transformation \(T\) provided \(T(P) = P\). A figure \(\mathcal{F}\) is invariant provided \(T(\mathcal{F}) = \mathcal{F}\); if \(T(P) = P\) for every \(P \in \mathcal{F}\) we say that the invariant figure \(\mathcal{F}\) is fixed point-wise.
A note on orientation in the plane.

Euclid rarely dealt with oriented figures but the work of Pappus relied upon them implicitly, and they are essential to the discussion of transformations. Orientation is an important concept - you worked with it in vector calculus when you used normal vectors to define the orientation of a surface. We can understand orientation in the Euclidean plane as a special case of that construction: *Is triangle $ABC$ the same as triangle $ACB$?* As figures they are the same collection of points but Pappus noted that if the direction from $A$ to $B$ to $C$ is, say, clockwise, the the direction from $A$ to $C$ to $B$ is counterclockwise. Thus they are different as oriented figures. The entire plane can be assigned an orientation relative to a given triangle $ABC$ and it would be assigned the opposite orientation relative to $ACB$. Our discussion of transformations depends on this distinction. Pappus gave the example of an isosceles triangle with $AB = AC$ and imagined a "reflection" of the plane that fixes $A$ and interchanges $B$ and $C$. Then triangle $ABC$ is mapped to triangle $ACB$, so the reflection reversed whatever orientation the plane was originally assigned. Most of Pappus’s contemporaries failed to appreciate what he was trying to do.

A note on the use of the word “transformation”.

In mathematics, a term often has a meaning specific to the topic of discussion. For example, in calculus or linear algebra a transformation may or may not be invertible, and may have a domain smaller than all of the points. In geometry, however, we always want transformations to be invertible, and to apply to the entire plane. So we will see that for any transformation $T$ and any given point $P$, there is a unique point $P' = T(P)$; and, given any point $Q$ there is a unique point $P$ such that $T(P) = Q$. 
Some Theorems from Euclid’s Elements

The first 28 propositions from Book I do not require Euclid’s Parallel Postulate:

**EPP.** If a transversal \(k\) intersects two lines \(l, m\) such that the sum of the (un-oriented) measures of the interior angles in one of the half-planes of \(k\) is less than two right angles, then \(l\) and \(m\) intersect in that half-plane.

I.1 Construct an equilateral triangle on a given segment.
I.2 Place at a given point a segment equal to a given segment.
I.3 Given two unequal segments, cut off from the greater a segment equal to the lesser.
I.4 SAS
I.5 Base angles of isosceles are equal (and exterior extensions).
I.6 Sides opposite equal base angles are equal.
I.7 Uniqueness of triangle with given base and sides in same half-plane relative to base.
I.8 SSS
I.9 Bisect an angle.
I.10 Bisect a segment.
I.11 Construct a \(\perp\) at a given point on a line.
I.12 Drop a \(\perp\) to a line from a given point.
I.13 Angles at intersection of two lines are supplementary.
I.14 Rays making supplementary angles at the vertex of a common ray between them are collinear.
I.15 Vertical angles are equal.
I.16 An exterior angle of a triangle is greater than either of the interior and opposite angles.
I.17 The sum of two interior angles in a triangle is less than two right angles.
I.18 In a triangle, the greater side subtends the greater angle.
I.19 Converse of I.18.
I.20 Strict triangle inequality.
I.21 If \(D\) is an interior point of \(\triangle ABC\) then \(BD + DC < BA + AC\) and \(\angle BDC > \angle BAC\).
I.22 Construct a triangle from three segments of given lengths satisfying I.20.
I.23 Construct an angle equal to a given angle at a given point on a given line.
I.24 If in \(\triangle ABC\) and \(\triangle DEF\) we have \(AB = DE\) and \(AC = DF\) and \(\angle BAC > \angle EDF\), then \(BC > EF\).
I.25 Converse of I.24 (if \(BC > EF\) then \(\angle BAC > \angle EDF\)).
I.26 ASA
I.27 Equal alternate angles implies parallel.
I.28 Equal opposite interior/exterior angles, or supplementary opposite interior angles, implies parallel.
I.29 Converges of I.27 and I.28 [existence of at least one parallel through a given point].
I.30 Lines parallel to a given line are parallel to each other.
I.31 Construct a parallel to a given line through a given point.
I.32 Exterior angle of a triangle is equal to sum of opposite interior angles and sum of interior angles is two right angles.
I.33 If \( AB = CD \) and \( \overline{AB} \parallel \overline{CD} \) then \( \overline{AC} = \overline{BD} \) and \( \overline{AC} \parallel \overline{BD} \).
I.34 Opposite side and angles of a parallelogram are equal and a diagonal divides the area in half.
I.35 Parallelograms with a common base and opposite side on the same line have equal areas.
I.36 Parallelograms with equal and collinear bases and opposite side on the same line have equal areas.
I.37 Triangles with a common base and third vertex on the line parallel to the base have equal areas.
I.38 Triangles with equal and collinear bases and third vertex on the line parallel to the base have equal areas.
I.39 If triangles \( ABC \) and \( DBC \) have equal areas then \( \overline{AD} \parallel \overline{BC} \).
I.40 If triangles \( ABC \) and \( CDE \) have equal areas with \( BC = CE \) and \( B, C, E \) collinear, then \( \overline{AD} \parallel \overline{BC} \).
I.41 If a parallelogram and triangle have the same base and the third vertex of the triangle is on the line through the opposite side of the parallelogram, then the area of the parallelogram is twice that of the triangle.
I.42 Construct a parallelogram with a given angle equal in area to a given triangle.
I.43 Given a parallelogram and a point on a diagonal, the two parallelograms determined by this point and the other two vertices have equal areas.
I.44 Construct a parallelogram with a given angle on a given segment as side with area equal to that of a given triangle.
I.45 Construct a parallelogram with a given angle equal in area to a given rectilineal figure.
I.46 Construct a square on a given segment.
I.47 Pythagorean Theorem.
I.48 Converse of Pythagorean Theorem.

II.12 Law of Cosines (obtuse angle).
II.13 Law of Cosines (acute angle).

III.1 Construct the center of a given circle.
III.3 A diameter bisects a chord iff it is perpendicular to it.
III.4 Two chords that are not diameters cannot bisect each other.
III.10 Two distinct circles have at most two points in common.
III.11 The line through the centers of tangent circles contains the tangent point (internal case).
III.12 The line through the centers of tangent circles contains the tangent point (external case).

III.13 A circle cannot be tangent to another circle at more than one point.

III.14 Equal chords are equally distant from the center, and conversely.

III.16 A perpendicular to a diameter at an endpoint falls outside the circle.

III.17 Construct a tangent to a circle from a given point.

III.18 A tangent is perpendicular to the diameter at the tangent point.

III.19 Converse of III.18.

III.20 A central angle with a given base arc is double a circumference angle with the same base arc.

III.21 Corollary of III.20 (equality of circumference angles).

III.22 Opposite angles of a quadrilateral inscribed in a circle sum to two right angles.

III.25 Construct the full circle from a given arc.

III.27 Generalization of III.21 to equal circles.

III.31 Thales’s Theorem and related angles.

III.32 The supplementary angles made by a tangent at the endpoint of a chord are respectively equal to the circumference angles on the opposite sides of the chord.

III.33 Given \( AB \), construct a circle that circumscribes a triangle \( ABC \), where the angle at \( C \) has also been given.

III.34 Given a circle, inscribe a triangle having a given angle.

III.35 If chords \( AC \) and \( BD \) intersect at \( E \) then \( AE \cdot EC = BE \cdot ED \).

III.36 If \( DB \) is tangent to a circle at \( B \) and \( DA \) cuts the circle at \( A \) and \( C \) then \( DA \cdot DC = DB \cdot DB \).

III.37 Converse of III.36 (equality implies \( DB \) is a tangent).

IV.4 Inscribe a circle in a given triangle.

IV.5 Circumscribe a circle about a given triangle.

IV.6 Inscribe a square in a given circle.

IV.7 Circumscribe a square about a given circle.

IV.8 Inscribe a circle in a given square.

IV.9 Circumscribe a circle about a given square.

IV.10 Construct an isosceles triangle with each base angle twice the vertex angle.

IV.11 Inscribe a regular pentagon in a given circle.

IV.12 Circumscribe a regular pentagon about a given circle.

IV.13 Inscribe a circle in a regular pentagon.

IV.14 Circumscribe a circle about a regular pentagon.

IV.15 Inscribe a regular hexagon in a given circle.

IV.16 Inscribe a regular 15-gon in a given circle.

VI.1 Triangles and parallelograms with equal heights have areas in proportion to their bases.

VI.2 A segment between two sides of triangle is parallel to the third side iff it divides the sides proportionally.
VI.3 In a triangle, a cevian divides the opposite side in the same ratio as the sides on the vertex iff it bisects the angle at that vertex.

VI.4 The corresponding sides of equiangular triangles are proportional.

VI.5 Converse of VI.4.

VI.8 The altitude from a right angle creates two triangles equiangular to the whole.

VI.13 Construct the mean proportional of two lengths.

VI.19 For equiangular triangles, the ratio of areas is the square of the ratio of corresponding sides.

VI.24 Given parallelogram $ABCD$ with diagonal $AC$, let $P, Q, R, S$ be on $AB, BC, CD, DA$ respectively, such that $T = PR \cap QS$ is on $AC$ with $PR \parallel DA \parallel BC$ and $QS \parallel AB \parallel CD$. Then the corresponding sides of parallelograms $ABCD, APTS, TQCR$ are in proportion.

VI.30 Divide a segment so that the whole is to the longer as the longer is to the shorter.

VI.31 Generalized Pythagorean Theorem for right triangles.

VI.33 Basis of radian measure.
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First Graded Assignment

Let $ABC$ be an isosceles Euclidean triangle with $AB = AC$ and $\angle BAC = \theta$. Let $M$ and $N$ be the midpoints of $AB$ and $AC$, respectively, and let $\alpha = \angle MON$ where $O$ is the intersection of the perpendicular bisectors of $AB$ and $AC$. Let $\beta = \angle BIC$ where $I$ is the intersection of the bisectors of the base angles.

1) Express $\alpha$ in terms of $\theta$.

2) Show that $\beta \in \left(\frac{\pi}{2}, \pi\right)$ and express $\beta$ in terms of $\theta$.

3) Find the relation between $\alpha$ and $\beta$.

4) Suppose $AB = AC = 2$. Find the distance between the circumcenter and the incenter of $ABC$. What is the condition on $\theta$ for the circumcenter to be closer to $A$ than the incenter?
Week 1. Some History

It had for a long time been evident to me that geometry can in no way be viewed, like arithmetic or combination theory, as a branch of mathematics; instead geometry relates to something already given in nature, namely, space.

(Hermann Grassmann, Die Ausdehnungslehre, 1844)

Is Euclid I.4 (SAS) actually the proof of a theorem?

Pappus and other geometers didn’t think so, nor did they believe Euclid’s proof of I.8 (SSS), because both proofs require that one triangle "be applied" to the other. Pappus produced alternative proofs for many of Euclid’s propositions by defining "congruence" to mean "relatable by rigid motion". His critiques were part of an inquiry that lasted for centuries and culminated in David Hilbert’s formulation (1899) of a complete and consistent set of axioms for Euclidean geometry, and the formalization of congruence in terms of transformations. One of Hilbert’s axioms (IV-6) is equivalent to SAS:

If triangles $ABC$ and $DEF$ are such that $AB = DE$, $BC = EF$ and $\angle B = \angle E$, then $CA = FD$, $\angle A = \angle D$ and $\angle C = \angle F$.

Note that this statement is in terms of measurements.

Hilbert’s axioms are not only consistent, they are complete. There are between fourteen and twenty-one of them depending on how you count, but we can pare them down to a few basics. For example, we accept without deeper justification some of Euclid’s "common notions" such as

If $C$ is on segment $\overline{AB}$ but is neither $A$ nor $B$ then $AB$ is greater than $AC$.

In triangle $ABC$, if $D$ is on segment $\overline{AC}$ but is neither $A$ nor $C$ then $\angle ABC$ is greater than $\angle ABE$ for any $E$ on $\overline{BD}$.

Euclid uses these as givens, together with $\text{SAS}$, to prove proposition I.16:
The exterior angle of a triangle is greater in measure than either of its opposite interior angles.

**Proof.** In triangle $ABC$ let $D$ be on $\overrightarrow{BC}$ but not on $\overline{BC}$. Let $E$ be the midpoint of $\overline{AC}$ and let $F$ be on $\overline{BE}$ such that $EF = BE$. Since $\angle AEB$ and $\angle CEF$ are vertical angles they are equal in measure, and so triangles $ABE$ and $CFE$ correspond by SAS. Therefore $\angle BAE = \angle FCE$, and since $\angle ECD > \angle ECF$ it follows that $\angle ACD > \angle FCE = \angle BAE = \angle BAC$. By a similar argument (bisecting $\overline{BC}$ instead), $\angle ACD > \angle ABC$. ■

The first twenty-eight propositions in Book I of *The Elements* do not require Euclid’s parallel postulate but many, such as I.16 and I.26 (ASA), use SAS.

Look at Euclid’s proof of I.26. It uses only SAS and the "common notions" above. (He also uses I.16 in the proof but does not really need it.)

**Exercise 1.** Proof of Euclid I.26.

Later we will see that, using transformations, we can get SSS from SAS.  
*We add SAS to Euclid’s five postulates:*

*Two distinct points determine a segment.*  
*Any segment extends uniquely and continuously to a straight line.*  
*There is a unique circle with given center and radius.*  
*All right angles are equal.*  
**SAS**  
**EPP** (*Euclid’s Parallel Postulate*)

Hilbert realized that Euclid’s second postulate defines a line as a subset of the points that comprise the plane. In order to make his axioms complete, Hilbert unpacked further assumptions hidden in Euclid’s postulates.

*What, for example, did Euclid mean by the fourth postulate?*
He was at once asserting that angles can be measured unambiguously, and that this measurement does not change with location. (Pappus noted that Euclid never defines congruence but refers to superposition of figures on occasion; thus, any right angle can be superimposed on any other right angle.) From this postulate we obtain the notion of complementary and supplementary angles, and find that vertical angles have equal measures.

\[
\begin{align*}
\alpha + \gamma &= 180^\circ = \beta + \gamma \\
\Rightarrow \alpha &= \beta
\end{align*}
\]

Euclid’s I.27 and I.28 are familiar from a first geometry course. As noted, they do not require EPP (Euclid’s parallel postulate). However, their converses (also familiar from a first course) do require it. Let’s sketch Euclid’s proof of the first claim in I.29.

Let \( \overrightarrow{EF} \) intersect the parallel lines \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \), as shown. Then \( \angle AGH = \angle GHD \).
**Proof.** Otherwise, one has greater measure, say, \( \angle AGH > \angle GHD \). Then

\[
180^\circ = \angle AGH + \angle BGH > \angle GHD + \angle BGH
\]

so the interior angles on the same side of \( \overrightarrow{EF} \) sum to less than two right angles. By EPP, \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) must intersect on that side, contradicting the hypothesis that \( AB \parallel CD \). □

**Exercise 2.** Determine whether Euclid’s I.30 requires EPP. What about I.31?

Proposition I.32 includes the assertion that the measures of the interior angles of a triangle sum to 180°. Later we will show that it is equivalent to EPP. The remaining propositions in Book I that follow I.32 all require EPP.

**Exercise 3.** Prove Euclid’s proposition I.33 and explain why it depends on EPP.

Propositions I.34 through I.45 establish properties of triangles and parallelograms needed for Euclidean area formulas.
Exercise 4. Prove Euclid’s proposition I.37 and explain why it depends on EPP.

Exercise 5. Proposition I.46 is a construction. Carry out the construction and explain where it uses EPP.

Book I concludes with the Pythagorean Theorem (proposition I.47, which uses SAS, I.14, I.41 and I.46) and its converse (proposition I.48, which uses SSS and I.47).

Hilbert’s work was motivated by the discovery of hyperbolic geometry, a type of non-Euclidean geometry that results when we abandon EPP and replace it with the postulate that there is more than one parallel through a point not on a given line. For the following exercises, assume EPP.

Exercise 6. The centroid $G$ of a triangle is the intersection of its medians, the circumcenter $O$ is the intersection of the perpendicular bisectors of the sides, and the orthocenter $H$ is the intersection of the altitudes. Show that $G, O, H$ are collinear.

Exercise 7. Let $ABC$ be isosceles with $AB = AC$, and let $M$ and $N$ be the midpoints of $AB$ and $AC$, respectively. Express $\angle NGM$ in terms of $\angle BAC$.

Exercise 8. The incenter $I$ of a triangle is the intersection of the internal angle bisectors. Let $ABC$ be isosceles with $AB = AC = s$ and $\theta = \angle BAC$. Find the radius of the inscribed circle in terms of $s$ and $\theta$. 
Week 2. Why We Need Transformations

Hilbert’s contemporary, Felix Klein, was motivated by the Pappus interpretation of congruence by rigid motion, so he defined “a geometry” by the transformations that preserve its properties.

A transformation (of the plane) is a 1-to-1 and onto mapping of the points. Thus, for any transformation $T$ there is an inverse transformation $T^{-1}$.

A collineation is a transformation that takes lines to lines.

If a unit of measure has been defined then we can talk about the collineations needed to establish congruence:

An isometry (rigid motion) is a collineation that preserves lengths of segments.

In contrast to Euclid’s common notions, Klein defined a figure to be any subset of the points of the plane. We say two figures are congruent provided there is a rigid motion that maps one onto the other. If $F_1$ and $F_2$ are two figures, $T$ is a rigid motion, and $T(F_1) = F_2$ then we write $F_1 \simeq F_2$. It follows that $T^{-1}(F_2) = F_1$.

Reflection is a Rigid Motion

What are these rigid motions that allow us to establish congruence of figures? Pappus understood that reflection in a line is the most fundamental step in producing superposition.

We still use his definition:

The reflection in line $l$ is the mapping that takes point $P$ to the point $P'$ such that $l$ is the perpendicular bisector of segment $PP'$.

Note that if $P$ is on $l$ then $P' = P$, whereas if $P$ is not on $l$ then $P'$ and $P$ are in opposite half-planes.

It follows that the fixed points of the reflection are precisely those on its axis $l$. The invariant line $l$ is fixed point-wise.

Any reflection $R$ is a transformation, in fact, $R^{-1} = R$. 
In fact, $R$ is a collineation: If $C$ is on $\overline{AB}$ then $C'$ is on $\overline{A'B'}$. The proof requires the Triangle Inequality, which does not depend on EPP (so it also holds in hyperbolic geometry). Euclid I.20 is the Triangle Inequality in its basic form; the proof uses only I.3 and I.19. We state the complete form which, in the spirit of Euclid, uses $PQ$ to mean the length of $\overline{PQ}$.

**Triangle Inequality.** For any three points $A, B, C$

$$AC + CB \geq AB$$

and $AC + CB = AB$ if and only if $C$ is on $\overline{AB}$.

**Theorem 2.1** Let $A'$ and $B'$ be the reflections of points $A$ and $B$, respectively, in a line $l$. Then $A'B' = AB$.

**Proof.** Let $P$ be the midpoint of $\overline{AA'}$ and $Q$ be the midpoint of $\overline{BB'}$. (So $P$ and $Q$ are on $l$.) Then triangles $APQ$ and $A'PQ$ correspond by SAS because $PQ$ is a common side, $A'P = AP$, and both angles at $P$ are right angles. Thus $AQ = A'Q$, $\angle PAQ = \angle P'A'Q$, and $\angle AQP = \angle A'QP$. Then, triangles $ABQ$ and $A'B'Q$ correspond by SAS because $AQ = A'Q$, $BQ = B'Q$, and $\angle AQB = 90^\circ - \angle AQP = 90^\circ - \angle A'QP = \angle A'B'Q$. Thus $AB = A'B'$ (and $\angle ABQ = \angle A'B'Q$, $\angle BAQ = \angle B'A'Q$).

**Corollary.** Any reflection is a collineation.

**Proof.** Without loss of generality we can assume $C$ is on $\overline{AB}$. We will show that $C'$ is on $\overline{A'B'}$. Let $P$ be the midpoint of $\overline{AA'}$, $Q$ be the midpoint of $\overline{BB'}$, and $O$ be the midpoint of $\overline{CC'}$. Then triangles $CPO$ and $C'PO$ correspond by SAS, so $CP = C'P$ and $\angle CPO = \angle C'PO$. But then the complements of these angles are equal, $\angle APC = \angle A'PC'$, so $A'C' = AC$, again by SAS. Similarly, triangles $CQO$ and $C'OQ$ correspond by SAS, and so $B'C' = BC$. Thus $AB = AC + CB = A'C' + C'B'$. However, by Theorem 2.1, $AB = A'B'$, and so the Triangle Inequality implies $C'$ is on $\overline{A'B'}$.

This proof has also shown that if the vertex of an angle is on $l$ (for example, $\angle AQB$) then its reflection in $l$ has the same measure.

More generally:

**Theorem 2.2** Reflection preserves the measure of angles.

**Proof.** Let $\angle A'B'C'$ be the reflection of $\angle ABC$ in $l$. We can assume $A, B, C$ are not collinear, for then each of our angles is $180^\circ$.
and the proof follows from the fact that reflection is a collineation. We can also assume there is a point $P = \ell \cap \overrightarrow{BC}$ (since at most one of the lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ is parallel to $\ell$). Let $D$ be the intersection of $\overrightarrow{AB}$ with the perpendicular to $\ell$ at $P$ (use $\overrightarrow{CA}$ if $\ell \parallel \overrightarrow{AB}$). Then $\angle BPD = \angle B'PD'$ because $P$ is on $\ell$ so, by SAS, $\angle DBP = \angle D'B'P$ because $BP = B'P$ (reflection is an isometry) and $DP = D'P$ (reflection is a collineation, so $D'$ is on $\overrightarrow{A'B'}$). But then $\angle ABC = \angle DBP = \angle D'B'P = \angle A'B'C'$. \hfill \blacksquare

In summary, by taking SAS as a postulate, we have shown that reflection in a line is a rigid motion, and then, as a corollary, that reflection is conformal (preserves measure of angles). Eventually we will determine all possible isometries of the Euclidean plane by showing that any rigid motion is a composition of reflections. There are conformal collineations that are not isometries. We will see, however, that any conformal collineation is the composition of a rigid motion and a dilation, which we will define later.

Application: Pappus’s Proofs of Euclid I.5 and I.6

The proposition that the base angles of an isosceles triangle have equal measure is usually proved in a first geometry course by constructing the bisector of the vertex angle and then applying SAS. Remarkably, Euclid did not use this simple proof, providing instead a proof that requires a relatively complicated construction. The converse statement is also typically proved in a first course, as an illustration of a proposition whose proof is not as straightforward as its converse. Pappus preferred to use symmetry to show that both proofs can be obtained by similar reasoning. We provide modern versions of his proofs using reflection as a mapping (which we have shown, using SAS, to be a rigid motion).

**I.5 The base angles of an isosceles triangle have equal measure.**

**Proof.** Let $ABC$ be a triangle with $AB = AC$ and let $\ell$ be the **angle bisector** at $A$. Upon reflection in $\ell$, $A \mapsto A$, and $B \mapsto B'$ with $B'$ on ray $\overrightarrow{AC}$ because reflection preserves angle measure. But it is given that $AB = AC$, so $B' = C$ because reflection preserves length. Similarly, $C' = B$, so $\angle ABC$ reflects to $\angle ACB$. Therefore these base angles have equal measure. \hfill \blacksquare

**Corollary.** $\ell$ is also the $\perp$-bisector of $BC$.

This proof shows that $\angle ABC \simeq \angle ACB$ because the reflection in $\ell$ takes one to the other. The figures $\angle ABC$ and $\angle ACB$ are congruent,
their measures $\angle ABC$ and $\angle ACB$ are equal.

**I.6** If two angles in a triangle have equal measure then their opposite sides have equal length.

**Proof.** Given that $\angle ABC = \angle ACB$, let $l$ be the perpendicular bisector of segment $BC$. Upon reflection in $l$, $ABC \rightarrow A'CB$ and so $\angle A'CB = \angle ABC = \angle ACB$. But then $A$ and $A'$ are both on ray $CA$. Similarly, $\angle A'BC = \angle ACB = \angle ABC$ and so $A$ and $A'$ are both on ray $BA$. Thus $A = A'$ and so $AB$ reflects to $AC$. Therefore $AB = AC$. □

**Corollary.** $l$ is also the bisector of $\angle BAC$.

This proof shows that $\overline{AB} \simeq \overline{AC}$ because the reflection in $l$ takes one to the other. The figures $\overline{AB}$ and $\overline{AC}$ are congruent, their measures $AB$ and $AC$ are equal.

As a corollary we get the following special case of SSS:

**Theorem 2.3** For distinct points $C$ and $D$, let triangles $ABC$ and $ABD$ be such that $AC = AD$ and $BC = BD$. Then the three pairs of corresponding angles are equal in measure.

**Proof.** If we construct segment $\overline{CD}$ we find, by I.5, that $\angle ACD = \angle ADC$ and that $\angle BCD = \angle BDC$. Then $\angle ACB = \angle ADB$, and so, by SAS the remaining pairs of corresponding angles are also equal. □

We write SSS* for Theorem 2.3.

**Corollary.** $\triangle ABC \simeq \triangle ABD$.

**Proof.** Using SAS between $\triangle AED$ and $\triangle AEC$, and between $\triangle BED$ and $\triangle BEC$ we see that $\overline{AB}$ is the $\perp$-bisector of $\overline{CD}$, so the reflection in $\overline{AB}$ interchanges $C$ and $D$. □

Some Related Theorems

Euclid III.3 can now be proved. It does not require the parallel postulate. (Book III of *The Elements* contains the theorems on circles typically encountered in a basic geometry course.)
III.3:

Exercise 9. If a diameter of a circle bisects a non-diameter chord then it is perpendicular to it.

Exercise 10. If a diameter of a circle is perpendicular to a non-diameter chord then it bisects it.

The first nineteen propositions of Book III do not require EPP.

Exercise 11. Prove Euclid III.20. Does it require EPP?

Exercise 12. Prove Euclid III.22. Show where EPP is required.

Exercise 13. Prove the converse of Euclid III.22.

Exercise 14. Suppose a circle can be inscribed in a given quadrilateral. Show that the sums of the lengths of opposite sides of the quadrilateral are equal.

Exercise 15. Suppose the sums of the lengths of opposite sides of a quadrilateral are equal. Show how to construct the inscribed circle.


Exercise 17. Prove Euclid III.35.


Exercise 20. Let $X$ be a point on the circumcircle of triangle $ABC$. Let $P$ be the closest point to $X$ on $BC$, $Q$ the closest point to $X$ on $CA$, and $R$ be the closest point to $X$ on $AB$. Show that $P, Q, R$ are collinear. (This is the Simson line for $X$.)
Week 3. Translations and Rotations

Using SAS we have shown that reflection is a rigid motion: Reflection preserves length and, as a consequence, also preserves measure of angle.

We can now make new rigid motions by composing reflections.

A translation is a mapping that moves every point a given distance in a given direction. (Thus, the translation is determined by $P \mapsto P'$, for any given point $P$. Any translation is a transformation.)

The rotation about point $O$ through angle $\alpha$ is the mapping $P \mapsto P'$ such that

(1) if point $P$ is different from $O$, then $OP = OP'$ and $\angle POP' = \alpha$; and

(2) if point $P$ is the same as point $O$, then $P' = P$.

(Thus, $P'$ is on the circle with center $O$ and radius $OP$. Any rotation is a transformation.)

**Theorem.** The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines through twice the distance between the lines.

**Theorem.** The composition of two reflections in lines that intersect at point $O$ is a rotation about $O$ through twice the angle between the lines.

We will prove these two theorems once we have discussed composition of transformations in more detail. We conclude from these theorems that translations and rotations are rigid motions, and use this conclusion to prove Euclid I.8:

**SSS.** If the sides of two triangles are correspondingly equal in pairs then the pairs of corresponding angles are equal in measure.

**Proof.** Let $ABC$ and $DEF$ be the two triangles with correspondingly equal sides. Apply the translation $A \mapsto A' = D$. Since translation is a rigid motion, we have $DB' = AB = DE$, so we next apply the rotation about $D$ that takes $B' \mapsto B'' = E$.

Where is $C''$, the image of $C$ after these two rigid motions?

On the one hand, it must be on the circle with center $D$ that passes through $F$ because $DF = AC = DC''$, but it must also be on the circle with center $E$ that passes through $F$ because $EF = BC = EC''$. So either $C'' = F$ (in which case $A \mapsto D, B \mapsto E, C \mapsto F$), or triangles $DEF$ and $DEC''$ correspond by SSS*. In the second
case, since \( \angle FDE = \angle C''DE \), line \( \overline{DE} \) is the bisector of \( \angle FDC'' \) so it is also the \( \perp \)-bisector of \( FC'' \) (Corollary to Euclid I.5); then reflection in \( \overline{DE} \) takes \( C'' \mapsto F \). In either case the corresponding angles are equal in pairs.

**Corollary.** \( ABC \simeq DEF \)

Overview of Transformations: General to Specific

**Mappings** - assign an image point to a given point

**Transformations** - invertible (one-to-one and onto) mappings

**Collineations** - transformations that take lines to lines

**Conformal Collineations** (Similarities) - collineations that preserve absolute angle measurement

**Isometries** (Rigid Motions) - collineations that preserve distance between points (isometries are conformal)
Equivalents of Euclid’s Parallel Postulate

There are many propositions in Euclidean plane geometry that are equivalent to \textit{EPP}. The equivalence of \textit{EPP} to the axiom of Proclus/Playfair/Hilbert (\textit{PPH}) and to the result that the sum of interior angles in any triangle is two right angles (\(\Delta_\pi\)) is shown below. We say that two lines are \textit{parallel} (\(l \parallel m\)) if there is no point in the plane that lies on both lines. (Euclid uses this word in his list of "common notions" but not in his postulates.) We will make use of Euclid’s propositions \textit{I.5, I.23, and I.27}. We also require \textit{Pasch’s Axiom}, which in one form states that a line through a vertex of a triangle, properly within the interior angle at that vertex, must intersect the segment opposite the vertex.

\textbf{EPP implies PPH}

Assuming \textit{EPP} we prove \textit{there is a unique line through a given point} \(X\), \textit{not on a given line} \(\overrightarrow{CD}\), \textit{that is parallel to} \(\overrightarrow{CD}\). Let \(Y\) be any point on \(\overrightarrow{CD}\) and let \(\overrightarrow{AB}\) be a line through \(X\) such that \(\angle DYX = \angle AXY\). Then \(\overrightarrow{AB} \parallel \overrightarrow{CD}\) by \textit{I.27}. If \(\overrightarrow{A'B'}\) is any line through \(X\) distinct from \(\overrightarrow{AB}\), with \(A'\) in the same half-plane as \(A\) (determined by \(\overrightarrow{XY}\)) and \(B'\) in the same half-plane as \(B\), then one of the following must be true:

\[
\angle A'XY < \angle AXY \\
\angle B'XY < \angle BXY
\]

In the first case, since \(\angle DYX = \pi - \angle CYX\) it follows that \(\angle A'XY + \angle CYX < \pi\). In the second case, since \(\angle CYX = \pi - \angle DYX\) it follows that \(\angle B'XY + \angle DYX < \pi\). In either case, \textit{EPP} implies that \(\overrightarrow{A'B'}\) must intersect \(\overrightarrow{CD}\). Thus \(\overrightarrow{AB}\) is the unique line through \(X\) that is parallel to \(\overrightarrow{CD}\).

\textbf{PPH implies} \(\Delta_\pi\)

Given triangle \(ABC\), let \(\overrightarrow{DE}\) be the unique line through \(A\) parallel to \(\overrightarrow{BC}\), with \(D\) and \(E\) chosen such that \(\angle BAD\) is alternate to \(\angle ABC\) and \(\angle EAC\) is alternate to \(\angle BCA\). Then \(\angle BAD = \angle ABC\) because any choice of point \(D'\) in the same half-plane as \(D\) (determined by \(\overrightarrow{AB}\)) such that \(\angle BAD' = \angle ABC\) (which exists by \textit{I.23}) implies \(\overrightarrow{AD'} \parallel \overrightarrow{BC}\) (by \textit{I.27}) and so, by \textit{PPH}, \(\overrightarrow{AD'}\) and \(\overrightarrow{DE}\) are the same line. Similarly, \(\angle EAC = \angle BCA\). It follows that \(\angle ABC + \angle BCA = \angle BAD + \angle EAC\). Since \(\angle BAD + \angle EAC + \angle CAB = \pi\) (straight angle) it follows that the sum of the interior angles of \(ABC\) also equals \(\pi\).
implies EPP

This last step requires Mathematical Induction, which helps explain why the independence of EPP was disputed for so many years.

Let \( \overline{AB} \) and \( \overline{CD} \) be distinct lines and let \( l \) be a transversal that intersects \( \overline{AB} \) at \( A \) and \( \overline{CD} \) at \( C \). Let \( \angle ACD = \alpha \) and \( \angle CAB = \beta \) and assume that \( \alpha + \beta < \pi \). We will inductively define a sequence of points \( \{D_n\} \):

Let \( D_1 = D \), let \( D_2 \) be the point on \( \overline{CD} \) such that \( D_1 \) is on \( \overline{CD}_2 \) with \( AD_1 = D_1D_2 \), and in general let \( D_{n+1} \) be the point on \( \overline{CD} \) such that \( D_n \) is on \( \overline{CD}_{n+1} \) with \( AD_n = D_nD_{n+1} \).

By construction, triangle \( AD_nD_{n+1} \) is isosceles with vertex \( D_n \). Let \( \delta_n = \angle AD_nC \). Then, for \( n > 1 \), we have \( \delta_{n-1} = \pi - \angle AD_{n-1}D_n \) and, by Euclid I.5, \( \delta_n = \angle D_nAD_{n-1} \). By hypothesis \((\Delta_x)\), the sum of the interior angles of any triangle equals \( \pi \), so \( \delta_{n-1} = \pi - (\pi - 2\delta_n) = 2\delta_n \). By induction we have

\[
\delta_n = \frac{1}{2^{n-1}} \delta_1
\]

Now let \( \beta_n = \angle CAD_n \). Again, by hypothesis,

\[
\beta_n = \pi - \alpha - \delta_n
\]

and so

\[
\lim_{n \to \infty} \beta_n = \pi - \alpha
\]

because

\[
\lim_{n \to \infty} \delta_n = 0
\]

But \( \pi - \alpha > \beta \) by construction, so for some large enough value of \( n \) we must have \( \beta_n > \beta \). Then \( \overline{AB} \) is properly within \( \angle CAD_n \) and so, by Pasch’s Axiom, \( \overline{AB} \) and \( \overline{CD} \) intersect in the half-plane of \( l \) where \( \alpha \) and \( \beta \) are measured. \( \blacksquare \)
Theorem 4.1 An isometry is determined by its action on any three non-collinear points.

Proof. Let $T$ be a rigid motion and let $A, B, C$ be non-collinear points, $T(A) = A', T(B) = B', T(C) = C'$. Given any point $X$ we must show that $T(X) = X'$ is determined by $A', B', C'$. Let $a = AX, b = BX, c = CX$. Let $C(P, r)$ be the circle with center $P$ and radius $r$. Then, by Euclid III.3, $C(A, a), C(B, b)$ and $C(C, c)$ have only $X$ in common because $A, B, C$ are non-collinear. Since $T$ is an isometry $X'$ must be on $C(A', a), C(B', b)$, and $C(C', c)$. Since $A, B, C$ are non-collinear so are $A', B', C'$ (otherwise $T^{-1}$ would not be a collineation). Thus $X'$ is the only point common to $C(A, a), C(B', b)$ and $C(C', c)$. It follows that $X' = T(X)$. ■

In fact, any collineation is determined by its action on any three non-collinear points. This result is known as the Fundamental Theorem of Affine Geometry (FTAG), which is equivalent to the following representation theorem, which we will discuss later:

Let $T$ be a collineation and represent any point $P$ by the ordered pair $(x, y)$ in $\mathbb{R}^2$. Then the coordinates of $T(P)$ are given by

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
+
\begin{pmatrix}
  p \\
  q
\end{pmatrix}
$$

for some choice of $a, b, c, d, p, q$ with $ad - bc \neq 0$.

We will not prove FTAG but we will see that the isometries of the plane are represented by the collineations with

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  a & c \\
  b & d
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}
$$

We will determine all rigid motions without using matrices or coordinates.
Transformations Generated by Two Reflections

We have already seen that a reflection $T$ in a line $l$ is an isometry. Thus the composition of any number of reflections is also an isometry. To keep notation as simple as possible we identify $T$ with the line $l$. If $m$ is another line we write

$$m \circ l$$

for the isometry obtained by first reflecting in $l$ and then in $m$. Thus, if $l = m$ (same line, which means same reflection) we have

$$m \circ l = l \circ m = I$$

where $I$ is the identity transformation, that is, a reflection is an involution (transformation equal to its inverse). Now suppose that $l$ and $m$ are distinct lines.

**Case 1):** $l \parallel m$

Consider $T = m \circ l$. (Then $l \circ m = T^{-1}$ because reflections are involutions.) For any point $P$, let $P' = l(P)$ and $P'' = T(P)$. Let $d$ be the oriented distance from $l$ to $m$ measured along their common perpendicular $n$ through $P$. Thus $d$ is the distance from $L$ to $M$, where $L$ is $l \cap n$ and $M$ is $m \cap n$.

**Notation.** If we write $d = LM$ then $ML = -d$.

Let $p = PL$, the distance from $P$ to $L$. Then $PP' = PL + LP' = 2p$. We know that $T(P)$ is on $n$ and so its location will be determined provided we can specify its distance from $P$. First,

$$PM = PL + LM = p + d$$

But $PM = PP' + P'M$ so

$$P'M = (d + p) - 2p = d - p$$

and $P'M + MP'' = P''P = 2(d - p)$ because $MP'' = P'M$. Finally,

$$PP'' = PP' + P'P''$$

$$= 2p + 2(d - p) = 2d$$

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Theorem 4.2 If \( l \parallel m \) then \( T = m \circ l \) is a translation. For any point \( P \) let \( n \) be the perpendicular to \( l \) and \( m \) through \( P \). Then \( T(P) \) is the point on \( n \) such that the distance from \( P \) to \( T(P) \) is twice the distance from \( l \) to \( m \). It follows that \( T^{-1} = l \circ m \), and that \( T = l_2 \circ l_1 \) for any two lines in the same parallel pencil such that the distance from \( l_1 \) to \( l_2 \) is the same as the distance from \( l \) to \( m \).

Composition of translations is commutative,
\[
T_1 \circ T_2 = T_2 \circ T_1
\]

We can think of composition of translations as addition of vectors. If \( l = m \) then \( T = I \), the identity transformation, which corresponds to the zero vector. Note that a non-identity translation has no fixed point, but any line parallel to the direction of translation is invariant.

Case 2): \( l \parallel m \)

Let \( O \) be the intersection of \( l \) and \( m \). We want to find \( T(P) \) for \( T = m \circ l \), relative to \( O \). Again, let \( P' = l(P) \) and \( P'' = T(P) \). Let \( \theta \) be the measure of the angle from \( l \) to \( m \), with the convention that \( \theta > 0 \) if measured in the counterclockwise direction. Given \( P \) different from \( O \) let \( C \) be the circle through \( P \) centered at \( O \) and let \( L \) be the midpoint of the segment \( PP' \). Note that \( P'' \) is on \( C \) because \( PP' \) is a chord with \( \perp \)-bisector \( l \). We will combine central angles of \( C \), observing the sign convention.

**Notation.** If we write \( \phi = \angle AOB \) then \( \angle BOA = -\phi \). With this convention we have
\[
\angle AOB + \angle BOC = \angle AOC
\]
Let \( \phi = \angle POL \). Then \( \angle POP' = \angle POL + \angle LOP' = 2\phi \). We know that \( T(P) \) is on \( C \) and so its location will be determined provided we can specify \( \angle POP'' \). Let \( M \) be the midpoint of the segment \( P'P'' \). Then
\[
\angle POM = \angle POL + \angle LOM = \phi + \theta
\]
But \( \angle POM = \angle POP'' + \angle P''OM \) so
\[
\angle P''OM = (\phi + \theta) - 2\phi = \theta - \phi
\]
and \( \angle P''OM + \angle MOP'' = \angle P''OP'' = 2(\theta - \phi) \) because \( \angle MOP'' = \angle P''OM \). Finally,
\[
\angle POP'' = \angle POP'' + \angle P''OP'' = 2\phi + 2(\theta - \phi) = 2\theta
\]
Theorem 4.3  If \( l \) and \( m \) intersect at \( O \) then \( T = m \circ l \) is a rotation. 
\( T(O) = O \), and for any point \( P \neq O \) the point \( P'' = T(P) \) is on the circle through \( P \) centered at \( O \) such that \( \angle POP'' = 2\theta \), where \( \theta \) is the angle measure from \( l \) to \( m \). It follows that \( T^{-1} = l \circ m \), so \( T^{-1} = T \) when \( \theta = \frac{\pi}{2} \) (half-turn). Also, \( T = l_2 \circ l_1 \), where \( l_1 \) and \( l_2 \) are any two lines in the pencil of lines concurrent at \( O \) such that the measure of the angle from \( l_1 \) to \( l_2 \) is equal to \( \theta \).

If \( \theta = 0 \) or \( \pi \) then \( l = m \) and \( T \) is the identity transformation \( I \). Note that a non-identity rotation has a unique fixed point, and no line is invariant unless \( \theta = \frac{\pi}{2} \), in which case every line through \( O \) is invariant. Any two rotations about \( O \) commute with each other, however, as we will see, rotations about distinct centers do not.
Suggested Exercises for Study

- Draw two parallel lines \( l \) and \( m \), and a third line \( n \) perpendicular to them. Let \( L = n \cap l \) and let \( M = n \cap m \), and assume the distance from \( L \) to \( M \) is 2 units. Let \( P \) be the midpoint of the segment \( LM \). Let \( P' = (m \circ l)(P) \).
  a) Locate \( P' \).
  b) Determine the oriented distance from \( M \) to \( P' \).
  c) Determine the oriented distance from \( L \) to \( P' \).

- Let \( l \) and \( m \) be lines through point \( O \) such that the angle from \( l \) to \( m \) is \( \frac{\pi}{2} \). Let \( T = l \circ m \).
  a) Determine the oriented angle from \( l \) to \( m \) (\( l \)).
  b) Determine the oriented angle from \( l \) to \( T \) (\( l \)).

- Draw two lines \( l \) and \( m \) through the point \( O \) such that the angle from \( l \) to \( m \) is \( \frac{\pi}{4} \). Locate a point \( P \) on the bisector of this angle such that the distance from \( O \) to \( P \) is 4 units. Let \( P_0 = (m \circ l)(P) \).
  a) Locate \( P_0 \).
  b) Determine the oriented angle from \( m \) to the line through \( OP_0 \).
  c) Determine the oriented angle from \( l \) to the line through \( OP_0 \).

- Let \( ABCD \) be a square. Label the vertices in this order counter-clockwise. Let \( T_1 \) be the reflection in line \( AB \) followed by the reflection in line \( AC \), and let \( T_2 \) be the reflection in line \( BD \) followed by the reflection in line \( CD \). Let \( T = T_2 \circ T_1 \).
  a) Determine \( T_1(B) \).
  b) Determine \( T_2(A) \).
  c) Determine \( T(X) \), where \( X = AC \cap BD \).
  d) Determine \( T(A), T(B), T(C), \) and \( T(D) \).
  e) Let \( P \) be an arbitrary point in the plane. Explain how to find \( T(P) \).

- Let \( ABC \) be an isosceles triangle with angle \( \angle ABC = \angle ACB \). Let \( b \) be the altitude through \( B \), let \( c \) be the altitude through \( C \), and let \( H = b \cap c \). Find \( \angle AHA' \) in terms of \( \angle BAC \), where \( A' = (c \circ b)(A) \).

- Let \( ABC \) be an isosceles triangle with angle \( \angle ABC = \angle ACB \). Let \( b \) be the internal angle bisector at \( B \), let \( c \) be the internal angle bisector at \( C \), and let \( I = b \cap c \). Find \( \angle AIA' \) in terms of \( \angle BAC \), where \( A' = (c \circ b)(A) \).

- Let \( ABC \) be an isosceles triangle with angle \( \angle ABC = \angle ACB \). Let \( b \) be the \( \perp \)-bisector of \( AC \), let \( c \) be the \( \perp \)-bisector of \( AB \), and let \( O = b \cap c \). Find \( \angle AOA' \) in terms of \( \angle BAC \), where \( A' = (c \circ b)(A) \).

- Let \( ABC \) be an equilateral triangle. Describe the isometry that takes
  a) triangle \( ABC \) to triangle \( ACB \)
b) triangle $ABC$ to triangle $BCA$

c) triangle $ABC$ to triangle $CAB$

d) triangle $ABC$ to triangle $CBA$

e) triangle $ABC$ to triangle $BAC$

f) triangle $ABC$ to triangle $ABC$

Let $ABCD$ be a square. Label the vertices in this order counter-clockwise. For which permutations of the vertices does there exist an isometry that takes $ABCD$ to the permuted vertices? Describe the isometry in each of these cases. How many are there?

Let $ABCD$ be a rhombus. Label the vertices in this order counter-clockwise and let $P = \overline{AC} \cap \overline{BD}$. Describe the isometry that takes

a) triangle $ABC$ to triangle $ADC$

b) triangle $ABD$ to triangle $CBD$.

c) triangle $APB$ to triangle $CPB$.

d) triangle $APB$ to triangle $APD$.

e) triangle $APB$ to triangle $CPD$.

f) If the rhombus is not a square, for which permutations of its vertices does there exist an isometry that takes $ABCD$ to the permuted vertices? Describe the isometry in each of these cases. How many are there?

In the coordinate plane, let $l$ be the line $y = x$ and let $m$ be the line $y = -x$. Let $AOB$ be the triangle with $A = (1, 0), O = (0, 0)$, and $B = (0, 1)$. Let $T = m \circ l$, and for any point $P$ let $P' = T(P)$.

a) Find the vertices of triangle $A'O'B'$.

b) Find the image of $AOB$ under $T^{-1}$.

c) Let $n$ be the line $y = x + 2$, and let $S = n \circ l$. Find the image of $AOB$ under $S$.

d) Find the image of $AOB$ under $S^{-1}$. 
Ceva’s Theorem (1678)

Let $ABC$ be any triangle and let $a, b, c$ be lines through $A, B, C$ respectively, each distinct from $\overrightarrow{AB}, \overrightarrow{BC},$ or $\overrightarrow{CA}$. Such lines are called cevians. **Ceva’s Theorem** provides a necessary and sufficient condition, in terms of the ratios in which they divide the opposite sides of the triangle, for these lines to be concurrent.

Let $P = a \cap \overrightarrow{BC}, Q = b \cap \overrightarrow{CA}, R = c \cap \overrightarrow{AB},$ and suppose $a, b, c$ are concurrent at $X$. For any triangle $UVW$ let $(UVW)$ denote its area. Consider triangles $PAB$ and $PCA$. We have, by Euclid VI.1

$$\frac{BP}{PC} = \frac{(PAB)}{(PCA)}$$

because both triangles have the same height when we consider both bases to be on $\overrightarrow{BC}$. (This step requires EPP.) For the same reason we also have

$$\frac{BP}{PC} = \frac{(PXB)}{(PCX)}$$

and so

$$\frac{BP}{PC} = \frac{(PAB) - (PXB)}{(PCA) - (PCX)} = \frac{(XAB)}{(XCA)}$$

*Here we have used the fact that if two ratios $\frac{p}{q}$ and $\frac{r}{s}$ are equal then each is equal to $\frac{p+tr}{q+ts}$, for any $t$.*

Similarly,

$$\frac{AR}{RB} = \frac{(XCA)}{(XBC)}$$

$$\frac{CQ}{QA} = \frac{(XBC)}{(XAB)}$$

Consequently

$$\frac{AR BP CQ}{RB PC QA} = \frac{(XCA) (XAB) (XBC)}{(XBC) (XCA) (XAB)} = 1$$

Conversely, if $\frac{AR BP CQ}{RB PC QA} = 1$ then the cevians must be concurrent because if $X$ is the intersection of any two of them, say $a \cap b = X$, with

$$\overrightarrow{CX} \cap \overrightarrow{AB} = R'$$

then

$$\frac{AR' BP CQ}{R'B PC QA} = 1 = \frac{AR BP CQ}{RB PC QA}$$

and so

$$\frac{AR'}{R'B} = \frac{AR}{RB}$$
But the ratio of distances to two given points uniquely determines the location of any point on the line through the given points. (This is the concept of affine coordinates discussed in Week 9.)

Thus, \( R = R' \), and we have proved Ceva’s Theorem:

**Theorem 4.4** Let \( a, b, c \) be cevians, respectively, of triangle \( ABC \), and let \( P = a \cap \overline{BC} \), \( Q = b \cap \overline{AC} \), \( R = c \cap \overline{AB} \). Then \( a, b, c \) are concurrent if and only if

\[
\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1
\]

Here are three familiar results, proved as corollaries to Ceva’s Theorem:

**Corollary.** The medians of a triangle are concurrent, the medians divide \( ABC \) into six triangles of equal area, and the medians trisect each other.

**Proof.** The medians are concurrent because \( \frac{AR}{RB} = \frac{BP}{PC} = \frac{CQ}{QA} = 1 \). Let \( G \) be the intersection of the medians (centroid). Next, let \( p = (BPG) = (GPC) \), \( q = (CQG) = (GQA) \), \( r = (ARG) = (GRB) \). Then \( 2p + r = (CRB) = (CAR) = 2q + r \) and so \( p = q \). Similarly, \( q = r \). Finally, \( (GAB) = 2(GBP) \) so, using a common altitude from \( B \), \( AG = 2GP \), and similarly, \( BG = 2GQ \) and \( CG = 2GR \).  

By considering the ratios to be signed quantities (taking orientation into account) the notation is consistent with signed areas, so Ceva’s Theorem is completely general in that it applies to cevians concurrent at a point exterior to the triangle as well.

**Corollary.** The altitudes of a triangle are concurrent (orthocenter).

**Proof.** Using the six right triangles, with \( BC = a, CA = b, AB = c \), we have the following equal ratios

\[
\frac{AR}{b} = \frac{QA}{c}, \quad \frac{RB}{a} = \frac{BP}{c}, \quad \frac{PC}{b} = \frac{CQ}{a}
\]
Then
\[
\begin{align*}
AR &= b \\
QA &= c \\
BP &= c \\
RB &= a \\
CQ &= a \\
PC &= b
\end{align*}
\]
and so
\[
\frac{AR \cdot BP \cdot CQ}{RB \cdot PC \cdot QA} = \frac{AR \cdot BP \cdot CQ}{QA \cdot RB \cdot PC} = 1
\]

**Corollary.** The $\perp$-bisectors of the sides of a triangle are concurrent (circumcenter).

**Proof.** The $\perp$-bisectors are the altitudes of the triangle whose vertices are the midpoints of the sides of the given triangle.

**Exercise 21.** Prove Euclid VI.3.

**Exercise 22.** Use Ceva’s Theorem, along with Euclid VI.3, to prove that the bisectors of the interior angles of a triangle are concurrent (incenter).

**Exercise 23.** Use Ceva’s Theorem to prove that the cevians through the contact points of the circle inscribed in a triangle are concurrent (Gergonne point).

Our proof of Ceva’s Theorem uses the adjacent/hypotenuse ratio that we now call the *cosine* of the included angle measure. Euclid II.12 and II.13 are equivalent to the Law of Cosines.

**Exercise 24.** Prove Euclid II.12.

**Exercise 25.** Prove Euclid II.13.

**Theorem of Menelaus**

Menelaus of Alexandria lived two centuries before Pappus of Alexandria and is associated with a theorem closely related to Ceva’s Theorem. The proof does not rely on ratios of areas, which perhaps explains why it was proved so much earlier. The following statement of Menelaus’s Theorem requires signed ratios.
Theorem 4.5: Given triangle $ABC$, and let $P \in \overrightarrow{BC}$, $Q \in \overrightarrow{CA}$, $R \in \overrightarrow{AB}$. Then $P, Q, R$ are collinear if and only if

$$\frac{AR \cdot BP \cdot CQ}{RB \cdot PC \cdot QA} = -1$$

Proof. Suppose $P, Q, R$ are on line $l$, and let $X, Y, Z$ be the feet of the perpendiculars to $l$ from $A, B, C$. In setting up signed ratios for the three corresponding right triangles, keep the orientations consistent. For example, $ARX \rightarrow BY$ so

$$\frac{AR}{RB} = \frac{XA}{BY}$$

Similarly,

$$\frac{BP}{PC} = \frac{YB}{CZ}$$
$$\frac{CQ}{QA} = \frac{ZC}{AX}$$

so $\frac{AR \cdot BP \cdot CQ}{RB \cdot PC \cdot QA} = (-1)^3 = -1$. Conversely, suppose $\frac{AR \cdot BP \cdot CQ}{RB \cdot PC \cdot QA} = -1$. Let $AB \cap PQ = R'$. Then $\frac{AR' \cdot BP \cdot CQ}{RB' \cdot PC \cdot QA} = -1$ and so

$$\frac{AR'}{RB'} = \frac{AR}{RB}$$

Thus $R' = R$ (see conclusion of the proof of Ceva’s Theorem), so $P, Q, R$ are collinear. \qed

Exercise 22’. The intersections of the bisectors of the exterior angles of a triangle with the sides opposite the vertices are collinear, as are the intersections of two interior-angle bisectors and the exterior bisector of the third angle with the opposite sides. What happens if the triangle is isosceles?
Week 5. Isometries of the Euclidean Plane, II

We have encountered three types of rigid motion: reflection, translation, rotation. The last two were obtained by composing two reflections, so to see if there are others we should experiment with composing additional reflections. It will turn out not to be necessary to consider the composition of more than three reflections.

Transformations Generated by Three Reflections

Let \( l, m, k \) be three distinct lines. We first consider two special cases.

**Case 1:** \( l, m, k \) are members of the same parallel pencil.

For any point \( P \), we need to determine \( T(P) \) where \( T = m \circ l \circ k \). By associativity, we can interpret \( T \) in two ways: the reflection \( k \) followed by the translation \( m \circ l \), or the translation \( l \circ k \) followed by the reflection \( m \). Choosing the first way, \( T = (m \circ l) \circ k \), let \( n \) be the common perpendicular through \( P \) and let \( K = k \cap n \). Let \( L = l \cap n \) and \( M = m \cap n \), with \( d \) the distance from \( L \) to \( M \). Let \( P' = k(P) \) and let \( P'' = T(P) \). If the distance from \( P \) to \( K \) is \( p \) then \( PP' = 2p \) and

\[
PP'' = PP' + P'P'' = 2p + 2d = 2(p + d)
\]

It follows that any point \( P \) moves along \( n \) through the distance \( 2(p + d) \). However, if \( J \) is the midpoint of \( PP'' \) then

\[
KJ = KP + PJ = -p + (p + d) = d
\]

so if \( j \) is the line in this pencil obtained by translating \( k \) through the distance \( d \), then \( j \) is the \( \bot \)-bisector of \( PP'' \). Thus, \( T(P) = j(P) \).

**Theorem 5.1** The composition of reflections in three distinct parallel lines is a reflection in another line of their parallel pencil; that line is the translation of the first line of reflection through the distance from the second line to the third line.

**Corollary.** If \( p = -d \) then \( PP'' = 0 \) and so \( T(P) = P \).
**Note.** You should show that $j$ is also the translation of the third line of reflection through the distance from the second line to the first line. **Suggestion:** Write $T = m \circ (l \circ k)$.

**Exercise 26.** Is it possible for $j$ to be $m$, $l$, or $k$? If so, how? If not, why not?

When either the first and second reflections, or the second and third reflections, are grouped together to produce a translation, their axes in the pencil are no longer relevant (Theorem 4.2). All that matters is that they are members of the pencil with the given oriented distance $d$ between them.

This makes it easy to remember which line to move in order to obtain $j$: Translate the remaining line of the pencil after grouping the other two for the translation.

**Case 2:** $l, m, k$ are concurrent.

For any point $P$, we need to determine $T(P)$ where $T = m\circ l\circ k$. Let $l \cap m \cap k = O$. As in **Case 1**, we can write $T = (m \circ l) \circ k$, the reflection $k$ followed by the rotation $m \circ l$. Given $P$, let $C$ be the circle through $P$ centered at $O$, and let $\theta$ be the measure of the angle from $l$ to $m$. Let $P' = k(P)$ and let $K$ be the midpoint of $PP''$. Let $P'' = T(P)$. If $\angle POP' = \phi$ then $\angle POP'' = 2\phi$ and $\angle P'OP'' = 2\theta$. Thus,

$$\angle POP'' = \angle POP' + \angle P'OP''$$

$$= 2(\phi + \theta)$$

However, if $J$ is the midpoint of $PP''$ then $\angle KOJ = \phi + \theta$ and

$$\angle KOJ = \angle KOP + \angle POJ = -\phi + (\phi + \theta) = \theta$$

so if $j$ is the line in this pencil obtained by rotating $k$ through the angle $\theta$, then $j$ is the bisector of $\angle POP''$. Thus, $T(P) = j(P)$.

**Theorem 5.2** The composition of reflections in three distinct concurrent lines is a reflection in another line of their concurrent pencil; that line is the rotation of the first line of reflection through the angle from the second line to the third line.

**Corollary.** If $\phi = -\theta$ then $\angle POP'' = 0$ and so $T(P) = P$.

**Note.** You should show that $j$ is also the rotation of the third line of reflection through the angle from the second line to the first line. **Suggestion:** Write $T = m \circ (l \circ k)$.
Exercise 27. Is it possible for $j$ to be $m$, $l$, or $k$? If so, how? If not, why not?

When either the first and second reflections, or the second and third reflections, are grouped together to produce a rotation, their axes in the pencil are no longer relevant (Theorem 4.3). All that matters is that they are lines through $O$ with the given oriented angle $\theta$ between them.

This makes it easy to remember which line to move in order to obtain $j$: Rotate the remaining line of the pencil after grouping the other two for the rotation.
Second Graded Assignment:
Modeling an Eclipse

Due: November 30

Let $D$ be the disk bounded by the unit circle $x^2 + y^2 = 1$ and let $D_t$ be the disk bounded by

$$(x - 2 + t)^2 + y^2 = 1$$

for $0 \leq t \leq 2$. Thus $D_2 = D$ and $D_0$ is tangent to $D$. 

$D_0$ and $D$ are tangent
Second Graded Assignment - Modeling an Eclipse

Let $A(t)$ be the area of $D_t \cap D$. Then $A(0) = 0$, $A(2) = \pi$, and $A$ is an increasing function on $[0, 2]$.

\[ D_t \text{ and } D \text{ overlap for } t > 0 \]

a) Find $A$ explicitly as a function of $t$. Graph this function on $[0, 2]$.

*Suggestion:* Let $P$ be an intersection of the two circles and look at triangle $OPO_t$ where $O = (0, 0)$ and $O_t$ is the center of $D_t$. Find $A$ as a function of $\theta = \angle OPO_t$ and then express $\theta$ as a function of $t$.

b) Compute the derivative $A'(t)$ and show that it is an increasing function on $(0, 2)$.

c) Compute the second derivative $A''(t)$. Find the (unique) value $t_0$ that satisfies the Mean Value Theorem for $A'(t)$ on $[0, 2]$. Sketch the tangent line to the graph of $A'(t)$ at the point $(t_0, A'(t_0))$.

d) Show that $1 + A(t_0)$ is half the area of $D$.

e) Extra Credit. Find the average value of $A(t)$ on $[0, 2]$. How does this value compare to half the area of $D$?
Three lines in general position

We say that three lines are in general position if their intersections are the vertices of a triangle. Suppose that \( l \cap m = O \) and that \( O \) is not on \( k \). Let \( T = m \circ l \circ k \) and let \( \theta \) be the measure of the angle from \( l \) to \( m \). Since the rotation \( m \circ l \) is the composition of reflections in any two lines through \( O \) where the angle from the first to the second is \( \theta \), there is no loss of generality in assuming \( l \parallel k \); then, when \( \theta = \frac{\pi}{2} \) the isometry \( T \) is called a glide. A glide has no fixed points but has a single invariant line (\( m \) in this case), called the axis of the glide. Apparently this is a new type of rigid motion.

**Note.** Suppose \( T = m \circ l \circ k \) is a glide with \( l \parallel k \). Show that \( T = l \circ k \circ m \), that is, the reflection \( m \) commutes with the translation \( l \circ k \).

The idea now is to show that, even if \( \theta \neq \frac{\pi}{2} \), the result is still a glide.

**Case 3:** \( T = m \circ l \circ k \) with \( l \parallel k \) and \( m \) a transversal.

The strategy is to express \( T = c \circ b \circ a \) with \( b \parallel c \) and \( a \) a perpendicular transversal:

- Let \( K \) be the intersection of \( k \) with its perpendicular through \( O \).
- Let \( a \) be the line through \( K \) that is parallel to \( m \).
- Let \( b \) be the line through \( K \) that is perpendicular to \( a \).
- Let \( c \) be the line through \( O \) that is perpendicular to \( a \).

Note that the measure of the angle from \( \overrightarrow{KO} \) to \( b \) is \( \theta \), as is the measure of the angle from \( \overrightarrow{KO} \) to \( c \), so, if \( d \) is the distance from \( k \) to \( l \), then the distance from \( a \) to \( m \) is \( d \cos \theta \) and the distance from \( b \) to \( c \) is \( d \sin \theta \).

Now let \( J = a \cap c \). If we can show that \( T' = c \circ b \circ a \) takes the vertices of the triangle \( K'OJ \) to the same images as \( T \) does then we have proved that \( T = T' \) (Theorem 4.1), that is, \( T \) is a glide. First, note that

\[
T(K) = (m \circ l)(K) \\
T'(K) = c(K)
\]

But the circle through \( K \) centered at \( O \) intersects \( a \) at \( K \) and its reflection in \( c \), because \( c \) is a diameter line of this circle. Thus \( c(K) = (m \circ l)(K) \) because \( m \circ l \) is rotation about \( O \) through \( 2 \theta \).
Next,
\[ k(O) = (b \circ a)(O) \]
because \(boa = H_K\), the half-turn about \(K\). But the circle through \(k(O)\) centered at \(O\) intersects the line through \(a(O)\) that is perpendicular to \(c\) at the reflection of \(k(O)\) in \(c\), so this intersection must also be the the rotation of \(k(O)\) through \(2\theta\) about \(O\). Thus \(T'(O) = T'(O)\).

Finally, let \(J' = T'(J)\), and note that \(J'\) is on \(a\), with \(JJ' = 2d\sin\theta\). Let \(J'' = m(J')\). Then \(J'J'' = 2d\cos\theta\), so the right triangle \(JJ''\) has hypotenuse of length \(2d\). But \(JJ''\) is perpendicular to \(l\) and \(k\) so \(J'' = (l \circ k)(J)\). It follows that \((m \circ l \circ k)(J) = (c \circ b \circ a)(J)\).

Thus \(T' = T\) because \(T(K) = T'(K)\), \(T(0) = T'(0)\), and \(T(J) = T'(J)\).

**Theorem 5.3** To determine the composition of reflections \(T = m \circ l \circ k\) in three lines in general position we can always assume \(l \parallel k\) and that \(m\) is a transversal. Let \(O = l \cap m\) and let \(K\) be the foot of the perpendicular through \(O\) to \(k\). Let \(a\) be the line through \(K\) parallel to \(m\), let \(b\) be the line through \(K\) perpendicular to \(a\), and let \(c\) be the line through \(O\) parallel to \(b\). Then
\[ T = (c \circ b) \circ a = a \circ (c \circ b) \]
so \(T\) is a glide, a reflection followed by a non-identity translation in a direction parallel to the line of the reflection.

**Notes.**

1) The reflection component of the glide commutes with the translation component. A glide has no fixed point and has a single invariant line, the *axis* of the glide.

2) As in the proof, let \(\theta\) be the measure of the angle from \(l\) to \(m\). The translation component of the glide moves every point in the direction \(\overrightarrow{KJ}\) through the distance \(2d\sin\theta\), where \(d = KO\).

3) \(T = c \circ (b \circ a) = (a \circ c) \circ b\), so we can also interpret the glide \(T\) as the half-turn about \(K\) followed by the reflection in \(c\), or as the reflection in \(b\) followed by the half-turn about \(J\).

4) \(T^{-1} = a \circ b \circ c = b \circ c \circ a\), so the inverse of the glide is obtained by reversing the direction of the translation component; equivalently, reflect in \(c\) and then perform the half-turn about \(K\); equivalently, perform the half-turn about \(J\) and then reflect in \(b\).

**Exercise 28.** Let \(k, l, m\) be the lines through the sides of an equilateral triangle \(ABC\). Let \(T = m \circ l \circ k\) and let \(P' = T(P)\). Find \(A'B'C'\).
Week 6. Composition of Finitely Many Reflections

We now show that increasing the number of reflections results in no new type of transformation, in other words, we have already found all rigid motions that can be produced by composing reflections. Suppose, then, that $T$ is a reflection, translation, rotation, or glide. We have seen that $T^{-1}$ is of the same type.

When we include a new reflection $l$ we can compose it in either order with $T$. But $T \circ l$ and $l \circ T^{-1}$ are inverses of each other, so it suffices to consider $l \circ T$ and to show it is one of our four types of isometry. Now, if $T$ is a reflection then $l \circ T$ is either a translation or rotation, whereas, if $T$ is a rotation or translation, we have seen that $T$ is either a reflection or a glide. Therefore, it suffices to suppose that $T$ is a glide and to analyze $l \circ T$. We assume $T = c \circ b \circ a$ with $a \bot b$ and $b \parallel c$, so if $l = c$ then $l \circ T$ is a half-turn. Otherwise,

**Case 1:** $l = a$ or $l = b$

If $l = a$ then $l \circ T = a \circ (c \circ b \circ a) = a \circ (a \circ c \circ b) = c \circ b$, the translation component of the glide $T$.

If $l = b$ then $l \circ T = (b \circ c \circ b) \circ a$. We have seen that $b \circ c \circ b = j$ where $j$ is the line obtained by translating $b$ through the distance from $c$ to $b$. Thus $l \circ T = j \circ a$ and $j \perp a$, so $l \circ T$ is the half-turn about $j \cap a$.

**Case 2:** $l \parallel a$

Let $O = a \cap b$ and let $Q = c \cap l$. let $H_O$ be the half-turn about $O$ and let $H_Q$ be the half-turn about $Q$. Then $l \circ T = (l \circ c) \circ (b \circ a) = H_Q \circ H_O$. For any point $P$ let $P' = H_O(P)$ and let $P'' = H_Q(P')$. Then $O$ is the midpoint of segment $PP''$ and $Q$ is the midpoint of segment $PP''$. Consider triangle $PP''P''$. We have $\overline{OQ} \parallel \overline{PP''}$ and $PP'' = 2OQ$. It follows that $l \circ T$ is the translation through twice the distance from $O$ to $Q$.

**Theorem 6.1** If $T$ is a glide and $l$ is parallel to the axis $a$ of the glide then $l \circ T$ is the composition of half-turns about two distinct points, resulting in a translation through twice the distance between these points in the direction from the first point to the second.

Note that this conclusion agrees with **Case 1** if we take $l = a$.

**Case 3:** $l \parallel a$
Let $Q = l \cap a$. Since $c \circ b$ is a translation we can assume that $c \cap a = Q$. Now $l \circ c \circ b \circ a = l \circ c \circ a \circ b$ because $b \circ a$ is a half-turn. We have seen that $l \circ c \circ a = j$, where $j$ is the line obtained by rotating $l$ about $Q$ through the angle from $c$ to $a$, so $j$ is the line through $Q$ that is perpendicular to $l$. Thus $T = j \circ b$ is a rotation about $J = j \cap b$. Note that the angle from $a$ to $l$ has the same measure as the angle from $b$ to $j$.

**Note.** If $K = a \cap b$ and we take $d = KQ$ to be positive then the length of $QJ$ is $d \csc \theta$, where $\theta$ is the measure of the angle from $a$ to $l$.

**Theorem 6.2** If $T$ is a glide and $l$ intersects the axis $a$ of the glide at $Q$ then $l \circ T$ is a rotation. The amount of rotation is $2\theta$, where $\theta$ is the measure of the angle from $a$ to $l$, and the center $J$ of the rotation is on the line through $Q$ that is perpendicular to $l$ whose distance from $Q$ is $d \csc \theta$ and such that $QKJ$ is a right triangle.

Note that the composition is a half-turn if and only if $l \perp a$.

**Exercise 29.** Let $l_1, l_2, l_3$ be the lines containing the sides of an equilateral triangle $ABC$, and let $l$ be one of the altitudes of the triangle. Let $T = l_3 \circ l_2 \circ l_1$ and let $P' = S(P)$, where $S = l \circ T$. Find $A'B'C'$. What type of rigid motion is $S$?

We have proved

**Theorem 6.3** The composition of finitely many reflections is either a reflection, translation, rotation, or glide. If the number of reflections is even (direct isometry) the result is either a translation or rotation; if the number of reflections is odd (opposite isometry) the result is either a reflection or glide.
Week 7. Classification of Isometries

The above theorem classifies the rigid motions of the plane that can be produced by a finite sequence of reflections. It remains to show that this accounts for all isometries of the plane. We will use the fact that any isometry is determined by its action on any three non-collinear points (Theorem 4.1). Let’s review the proof of this fact:

Let $ABC$ be a triangle and $T$ be an isometry with $T(A) = A'$, $T(B) = B'$, $T(C) = C'$. We have seen that if two circles intersect in two distinct points then the perpendicular bisector of their common chord contains the centers of both circles. Thus, for any point $P$, the circle centered at $A$ of radius $AP$, the circle centered at $B$ of radius $BP$, and the circle centered at $C$ of radius $CP$ intersect only at $P$ (otherwise $A, B, C$ would be collinear). Similarly, the circle centered at $A'$ of radius $AP$, the circle centered at $B'$ of radius $BP$, and the circle centered at $C'$ of radius $CP$ intersect at a single point, which must be $P' = T(P)$ since $AP = A'P'$, $BP = B'P'$, $CP = C'P'$.

Classification of Rigid Motions for the Euclidean Plane

Any isometry $T$ of the plane is either a reflection, translation, rotation, or glide. Further, $T$ can be produced by a sequence of three or fewer reflections.

Proof. Let $T$ be an isometry and let $ABC$ be a triangle with $T(A) = A'$, $T(B) = B'$, $T(C) = C'$.

Case 1: Suppose triangles $ABC$ and $A'B'C'$ have the same orientation. Let $T_1$ be the translation $A \mapsto A'$ and let $T_2$ be the rotation about $A'$ such that $(T_2 \circ T_1)(B) = B'$. The circle centered at $A'$ of radius $AC$ and circle centered at $B'$ of radius $BC$ intersect at $C'$ and its reflection in $A'B'$. Since translation and rotation preserve orientation we must have $(T_2 \circ T_1)(C) = C'$. Now $T$ is determined by the images $A', B', C'$ and so $T = T_2 \circ T_1$, which is itself either a translation or rotation.

Case 2: Suppose triangles $ABC$ and $A'B'C'$ have opposite orientation. Let $T_3$ be the reflection in $A'B'$. With $T_1$ and $T_2$ as in Case 1 we note that $T_3 \circ T_2 \circ T_1$ reverses orientation, so $(T_3 \circ T_2 \circ T_1)(C) = C'$. Thus $T = T_3 \circ T_2 \circ T_1$ and $T$ is either a reflection or a glide.
Exercise 30. In the coordinate plane, exhibit triangles $ABC$ and $A'B'C'$ with six distinct vertices and corresponding sides of equal length such that

i) $T : ABC \rightarrow A'B'C'$ is a translation in a direction not parallel to either coordinate axis

ii) $T : ABC \rightarrow A'B'C'$ is a rotation about a point other than the origin

iii) $T : ABC \rightarrow A'B'C'$ is a reflection in a line not parallel to either coordinate axis

iv) $T : ABC \rightarrow A'B'C'$ is a glide with axis not parallel to either coordinate axis
Suggested Exercises for Study

- Draw two parallel lines \( l \) and \( m \) and a third line \( k \) perpendicular to them. Let \( L = k \cap l \) and let \( M = k \cap m \), and let \( LM = 2 \). Let \( P \) be the midpoint of \( LM \).
  a) Let \( P' = (m \circ l)(P) \). Locate \( P' \) and determine the oriented distance \( MP' \).
  b) Let \( P'' = (l \circ m \circ l)(P) \). Locate \( P'' \) and determine the oriented distance \( P''P \).
  c) Find the line \( j \) such that \( P'' = j(P) \).

- Draw two lines \( l \) and \( m \) through a point \( O \) such that the measure of the angle from \( l \) to \( m \) is \( 3 \). Locate a point \( P \) on the bisector of this angle such that \( OP = 2 \).
  a) Let \( P_0 = (m \circ l)(P) \). Locate \( P_0 \) and determine the measure of the angle from \( m \) to \( OP_0 \).
  b) Let \( P_{00} = (l \circ m \circ l)(P) \). Locate \( P_{00} \) and determine the measure of the angle from \( OP_0 \) to \( OP_{00} \).
  c) Find the line \( j \) such that \( P_{00} = j(P) \).

- Let \( ABC \) be an equilateral triangle with side length 2, and let \( a, b, c \) be \( \overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB} \), respectively.
  a) Find the invariant line of the isometry \( T = c \circ b \circ a \).
  b) Represent \( ABC \) in the coordinate plane so that the origin is the foot of the perpendicular from \( A \) to \( a \) and \( C \) is on the positive \( x \)-axis. Find the coordinates of \( T(A), T(B), T(C) \).

- In the coordinate plane, let \( k, l, m, n \) be the lines \( y = 0, x = 0, x = 1, y = 1 \), respectively. Determine each of the following transformations \( T \), and find the image of the triangle \( AOB \), where \( A = (1, 0), O = (0, 0), \) and \( B = (0, 1) \).
  a) \( T = k \circ l \circ m \circ n \)
  b) \( T = m \circ n \circ k \circ l \)
  c) \( T = m \circ l \circ k \circ n \)
  d) \( T = n \circ k \circ l \circ m \)

- Let \( P \) and \( Q \) be distinct points. Let \( T_1 \) be the rotation about \( P \) through \( \theta = \frac{\pi}{2} \) and let \( T_2 \) be the translation from \( P \) to \( Q \).
  a) Show that the isometry \( T = T_2 \circ T_1 \) is a rotation.
  b) Where is the center of the rotation \( T \) ?
  c) Determine \( T \) if \( T_1 \) is the rotation about \( P \) for any given \( \theta \neq 0 \).

- Let \( P \) be a point on line \( l \) and let \( T \) be the rotation through \( \theta \) about \( P \).
  a) Determine the isometry \( T \circ l \).
  b) Determine the isometry \( l \circ T \circ l \).
Label the vertices of triangle $ABC$ counter-clockwise and measure the internal angles at $A, B, C$ counter-clockwise. Let $a, b, c$ be $BC, CA, AB$, respectively. For each vertex $V$, let $T_V$ be the rotation about $V$ through twice the internal angle measure at $V$.

a) Describe each of $T_A, T_B, T_C$ in terms of the reflections $a, b, c$.

b) Determine the isometry $T_B \circ T_C \circ T_A$.

c) What happens with other orders of composition of $T_A, T_B, T_C$?

For triangle $ABC$ let $a, b, c$ be the $\perp$-bisectors of $BC, CA, AB$, respectively. Determine the lines $k, l, m$ such that

a) $k = c \circ b \circ a$

b) $l = b \circ a \circ c$

c) $m = a \circ c \circ b$

d) There are six ways to order the reflections $a, b, c$. What happens with the other three orders of composition?

Let $P$ and $Q$ be distinct points on the line $l$. Let $H_P$ be the half-turn about $P$ and let $H_Q$ be the half-turn about $Q$.

a) Describe the isometry $T = H_Q \circ l \circ H_P$.

b) Find the invariant line of $T$.

c) Locate $T(M)$, where $M$ is the midpoint of $PQ$.

The composition of reflections in the three angle bisectors of a triangle is a reflection in some line through the incenter of the triangle. Determine this line.

Let $ABC$ be an isosceles triangle with $AB = AC$ and $\angle BAC = \frac{\pi}{6}$. Let $a, b, c$ be the altitudes from $A, B, C$, respectively.

a) Find $l$ such that $l = c \circ b \circ a$.

b) Let $P = l \cap BC$. Find $\angle HCP$ and $\angle HPC$ where $H$ is the orthocenter of $ABC$.

c) For what value of $\theta = \angle BAC$ would $l$ parallel to $BC$? Is there a value of $\theta$ for which $l$ is parallel either $\overrightarrow{AC}$ or $\overrightarrow{AB}$?

Let $C_1$ and $C_2$ be circles that intersect at two points $X$ and $Y$. Let $H_X$ be the half-turn about $X$. Then $C_2$ intersects $H_X(C_1)$ at $X$ and another point $Z$.

a) Locate $U = H_X(Z)$.

b) Locate $V = H_X(Y)$.

c) What type of quadrilateral is $YUVZ$?

d) Show that $YUVZ$ is a rhombus if $C_1$ and $C_2$ have the same radius.

e) If $C_1$ and $C_2$ both have radius $r$ and $YUVZ$ is a square, what is the distance between their centers?

For any triangle $ABC$, choose a point $P$ on $\overrightarrow{BC}$. Let $Q$ be the point on $\overrightarrow{CA}$ such that $\overrightarrow{PQ} \parallel \overrightarrow{AB}$, and let $R$ be the point on $\overrightarrow{AB}$ such that $\overrightarrow{QR} \parallel \overrightarrow{BC}$. For what choice of $P$ will the cevians $\overrightarrow{AP}, \overrightarrow{BQ}, \overrightarrow{CR}$ be concurrent?
Approximate Schedule of Remaining Topics

November 14
   Inversion in a circle and dilation; similarity transformations

November 16
   Fundamental Theorem of Affine Geometry

November 21
   Collineations in matrix/vector form

November 28
   Finding fixed points and invariant lines

November 30
   Transformation of general figures; similarity of all parabolas
Week 8. Similarity Transformations

We have defined two figures to be congruent provided there is an isometry that relates one to the other. In Euclidean geometry we can also define what it means for two figures to be similar.

Definition. Two figures are similar provided there is a conformal collineation that takes one to the other.

Recall that a transformation is conformal if it preserves absolute angle measure. Since rigid motions are conformal they are special cases of conformal collineations. Just as isometries are called rigid motions, conformal collineations are called similarities. The following theorem is a corollary of the Fundamental Theorem of Affine Geometry (which, you will recall, we do not prove in this course).

Theorem 8.1 Any similarity of the Euclidean plane is the composition of a rigid motion and a central dilation.

Definition. The (central) dilation about point $O$ with factor $k \neq 0$ is the transformation $P \mapsto P'$ such that $\overrightarrow{OP} = \overrightarrow{OP'}$ and $OP' = kOP$.

Notes.

1) If $k < 0$ the dilation is the composition of the dilation with factor $|k|$ followed by the half-turn about $O$. For this reason the dilation factor is often defined to be a positive number. We will refer to dilations with $k > 0$ as proper dilations.

2) Dilation is a conformal collineation that preserves orientation, and the image of any line $l$ is a line parallel to $l$.

Dilation and Inversion

Dilation is closely related to translation in that the image of any line $l$ is a line parallel to $l$. In fact, just as a translation can be factored into two reflections, a dilation can be factored into two inversions.

Definition. The inversion in the circle with center $Q$ and radius $r$ is the mapping $P \mapsto P'$, for $P \neq O$, such that $OP' = \overrightarrow{OP}$ and $(OP')(OP) = r^2$.

Note. Inversion is a transformation of the plane with the point $O$ removed. It is an involution on this so-called ‘punctured plane’. Note that $P = P'$ if and only if $P$ is on the circle of inversion.
Since circles may have different centers it is not convenient to work with various punctured planes. To avoid this situation we sometimes consider all lines to be circles that are concurrent at a point that is not in the Euclidean plane. When this point, commonly denoted by the symbol $\infty$, is added to the plane we obtain the \textit{inversive plane}.

\textbf{Theorem 8.2} Inversion is a conformal involution of the inversive plane that reverses orientation. (If $O$ is the center of the circle of inversion then $O' = \infty$.) Any line through the center of the circle of inversion is invariant. The image of any other line is a circle through $O$, and the image of any circle not through $O$ is another circle.

The proof of this theorem is not difficult. It is usually given in a course on inversive geometry. (Using Klein’s definition, the geometry of the inversive plane is described by the transformations that can be obtained by composing inversions. These transformations are collineations of the inversive plane because Euclidean lines and circles are equivalent figures. Note, however, that these so-called \textit{inversive lines} can intersect in more than one point and two points lie on infinitely many inversive lines.) For the purposes of this course we will show how inversions can be combined to produce proper dilations.

\textbf{Theorem 8.3} The composition of inversions in two concentric circles is a dilation about their center.

\textbf{Proof.} Let $C_1$ with radius $r_1$ and $C_2$ with radius $r_2$ be centered at $O$. Let $P$ be any point of the Euclidean plane and let $P'$ be the inversion of $P$ in $C_1$. If $P \neq O$ then $P'$ is on $OP$ with $(OP)(OP') = r_1^2$. If $P = O$ let $P' = \infty$. Let $P''$ be the inversion of $P'$ in $C_2$. If $P' \neq \infty$ then $P''$ is on $OP' = OP$ with $(OP')(OP'') = r_2^2$. If $P' = \infty$ let $P'' = O$. Thus, if $P = O$ then inversion of $P$ in $C_1$ followed by inversion in $C_2$ is just $O$. Otherwise

\[ \frac{OP''}{OP} = \frac{r_2^2}{r_1^2} \]

so in any case

\[ OP'' = \left( \frac{r_2}{r_1} \right)^2 OP \]

Therefore, inversion in $C_1$ followed by inversion in $C_2$ is the transformation of the Euclidean plane that is dilation about $O$ with factor $k = \left( \frac{r_2}{r_1} \right)^2$. \hfill \blacksquare
Exercise 31. The power of the point $P$ with respect to the circle $C$, with center $O$ and radius $r$, is

$$p = d^2 - r^2$$

where $d = OP$. In the coordinate plane, let $C$ be the circle with center $(a, b)$ and radius $r$. Let $P = (x, y)$. Show that $P'$, the inversion of $P$ in $C$, is

$$\left(\frac{r^2x + pa}{d^2}, \frac{r^2y + pb}{d^2}\right)$$

Thus, if $C$ is the unit circle then the coordinates of $P'$ are $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.

Exercise 32. In the coordinate plane, let $O = (0, 0), A = (1, 0), B = (0, 1)$. Let $T$ be the collineation $P \mapsto P'$ such that $O' = (1, 1), A' = (2, 2), B' = (0, 2)$. Express $T$ as a rotation followed by a dilation, and then as a dilation followed by a rotation. Show how to express each dilation as the composition of inversions in concentric circles.

Application: Circles of Apollonius

Given two distinct points $A$ and $B$, the locus of points $P$ in the plane such that $PA = PB$ is the perpendicular bisector of $AB$. Apollonius of Perga, an astronomer and geometer credited with many theorems on conics and related locus constructions, solved the following generalization of this result.

Given two distinct points $A$ and $B$, find the locus of points $P$ in the plane such that $PA = kPB$, where $k > 0$.

The construction due to Apollonius can be explained in terms of inversion. Working with absolute lengths, let $AB = BA = d$ and let $C$ be the circle centered at $A$ of radius $d$. For any point $P \neq A$ let $P'$ be its inversion in $C$. Since $B$ is on $C$ we have $B' = B$, so for any other point $P$ we have

$$\frac{(AP)(AP')}{(AB)} = \frac{d^2 = (AB)^2}{\frac{AP}{AP'}} = \frac{AP}{AB}$$

Since triangles $APB$ and $ABP'$ share the angle at $A$ it follows from Euclid VI.4 and VI.5 that all corresponding angles are equal in measure, and that

$$\frac{BP'}{PB} = \frac{AP'}{AB} = \frac{d^2}{(AP)(AB)}$$
Now suppose \( PA = kPB \). Then
\[
\frac{BP'}{PB} = \frac{d^2}{(kPB)(AB)}
\]
\[
BP' = \frac{d^2}{kAB} = \frac{d}{k}
\]

Thus, Apollonius concluded that \( P' \) is on a circle of radius \( \frac{d}{k} \) centered at \( B \).

Since \( P \) is on the inversion of this circle in \( C \), it follows that the locus of all such points \( P \) is a circle with center on \( AB \). If this circle intersects \( AB \) at \( C \) and \( D \) then its center is the midpoint of \( CD \). For a given diameter, there are exactly two pairs of points \( C \) and \( D \), such that the product of signed ratios
\[
\frac{AC}{CB} \cdot \frac{DB}{AD} = -1
\]
(This is the cross-ratio of \( A, B, C, D \). When this cross-ratio equals \(-1\) the points \( C \) and \( D \) are said to be in harmonic relation to \( A \) and \( B \).) For one pair the circle encloses \( B \) and for the other the circle encloses \( A \). For a given \( k \), Apollonius found the radius of the circle to be
\[
\frac{k}{d|k^2 - 1|}
\]
so the radius for \( k \) is the same as the radius for \( \frac{1}{k} \), as would be expected since \( PA = kPB \iff PB = \frac{1}{k}PA \): If \( k > 1 \) the circle encloses \( B \) and the circle for \( \frac{1}{k} \) encloses \( A \).
Exercise. In the coordinate plane, let $A = (-1, 0)$ and $B = (1, 0)$ and let $C_k$ be the locus of points $P$ such that $PA = kPB$. Find the center and radius of $C_k$. 
Week 9. Fundamental Theorem of Affine Geometry

While similarities are the most important transformations for Euclidean geometry (since we use them to define similar figures in general and congruent figures in particular), they are special cases of collineations of the plane. The representation of collineations by vectors or coordinates is essential in calculus and analytic geometry. We will return to that representation, but remember that it is equivalent to the following statement.

**FTAG.** Given two triangles $ABC$ and $A'B'C'$, the correspondence $ABC \rightarrow A'B'C'$ determines a collineation $T$. Any collineation can be described by such a correspondence.

A constructive proof of FTAG is not as easy to achieve as the proof we gave for rigid motions. Such a proof requires the theory of the affine plane, which depends on the following assignment of affine coordinates to a given line $l$.

Given two distinct points $A, B$ on the line $l$ there is a unique point $P$ on $l$ with the ratio $\frac{AP}{PB}$.

These are signed ratios as usual, so every real number corresponds to a point on $l$, but if $P = B$ then we must assign it a new coordinate, usually denoted $\infty$. The assignment of coordinates on $l$ is independent of the assignment on another line, so the affine plane does not work with absolute measurements, just ratios. The following proposition is equivalent to FTAG:

Any collineation $T$ of the affine plane preserves ratios on any given line.

If we take this proposition as given, the proof of FTAG is easy:

**Proof.** Given triangle $ABC$, any line $l$ must intersect at least two of the lines through its sides (because it can be parallel to at most one side). Suppose $l \cap \overrightarrow{AB} = R$ and $l \cap \overrightarrow{AC} = Q$. Now $R'$ is on $\overrightarrow{A'B'}$ because $T$ is a collineation. By hypothesis we must have $\frac{AR'}{PR'} = \frac{AR}{PB}$, so the location of $R'$ on $\overrightarrow{A'B'}$ is determined. Similarly, the location of $Q'$ on $\overrightarrow{A'C'}$ is determined because $\frac{C'Q'}{Q'A'} = \frac{CQ}{QA}$. Thus $T(l)$ is the line through $Q'$ and $R'$. $\blacksquare$
FTAG implies that not all collineations are similarity transformations. (In general, collineations are not conformal.) As an example, given an isosceles triangle \(ABC\) with right angle at \(B\), and let \(T\) be the collineation \(P \mapsto P'\) determined by \(A' = A, B' = B,\) and \(C'\) the point with \(C'A \perp AB\) and \(C'A = CB\) such that the image triangle has the same orientation as \(ABC\).

Given \(P\), where is \(P'\)?

The image point \(P'\) is on the line through \(P\) parallel to \(AB\) such that the length of \(PP'\) is equal to the length of \(PD\), where \(D\) is the foot of the perpendicular to \(AB\) through \(P\).

**Note.** Such a collineation is called a *shear*. It follows that any line parallel to \(AB\) is invariant and the fixed points are precisely those on \(AB\).

*How do we know this?* We could verify the claim using the preservation of ratios, but for our purposes it will be useful to return to the representation theorem for collineations that was mentioned as a generalization of Theorem 4.1.

**Theorem 9.1** Let \(T\) be a collineation and represent any point \(P\) by an ordered pair \((x, y)\) in \(\mathbb{R}^2\). Then the coordinates of \(T(P)\) are given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}
\]

for some choice of \(a, b, c, d, p, q\) with \(ad - bc \neq 0\).

Note that we use either \((x, y)\) or \(\begin{pmatrix} x \\ y \end{pmatrix}\) as context requires.

For the shear \(T\) in our example, suppose \(A = (-1, 1), B = (-2, 1), C = (-2, 2)\) with \(A' = A\) and \(B' = B\). Then \(C' = (-1, 2)\), so we must have

\[
\begin{align*}
-a + b + p &= -1 \\
-c + d + q &= 1 \\
-2a + b + p &= -2 \\
-2c + d + q &= 1 \\
-2a + 2b + p &= -1 \\
-2c + 2d + q &= 2
\end{align*}
\]

The system is easily solved (the equations always decouple into two sets with three variables each) to find

\[
a = 1, b = 1, c = 0, d = 1, p = -1, q = 0
\]
Thus
\[
T(x, y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} x + y - 1 \\ y \end{pmatrix}
\]

**Exercise 33.** Verify the claim: For any point \( P \), \( P' = T(P) \) is on the line through \( P \) parallel to \( \overline{AB} \) such that the length of \( \overline{PP'} \) is equal to the length of \( \overline{PD} \), where \( D \) is the foot of the perpendicular to \( \overline{AB} \) through \( P \).

**Exercise 34.** Let \( DEF \) be the isosceles triangle \( D = (-1, 0) \), \( E = (1, 0) \), \( F = (0, 1) \). Sketch \( D'E'F' \). This example shows the importance of using isometries to define congruence. Triangles \( ABC \) and \( A'B'C' \) may "appear" to be congruent, but they are not \( (A'C' \neq AC) \). The collineation they determine is not a rigid motion of the plane.

This analytic representation of a collineation \( T \) allows us to describe it in terms of the chosen coordinates, even when the geometric properties of its action on the plane are not obvious.

**Example.** Let \( T(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \). This is a collineation because \( ad - bc \neq 0 \). If \( O \) is the origin then \( O' \) has coordinates \( (0, 3) \). If \( (x, y) \) is a fixed point then
\[
2x + y = x \\
x + y + 3 = y
\]
which has the unique solution \( x = -3, y = 3 \) so the only fixed point is \( (-3, 3) \).

**Exercise 35.** Any line in the plane is described by an equation of the form
\[
Ax + By = C
\]
with \( A^2 + B^2 \neq 0 \). Show that the collineation \( T \) in the example maps this line to the line
\[
(B - 2A)x + (3A - 2B)y = 3B - 5A - C
\]
Similarities in Analytic Form

If a collineation \( T \) happens to be a similarity transformation we should be able to recognize that fact from its representation

\[
T(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}
\]

First, note that the representation of any collineation has two parts - a matrix action (sometimes called the linear component) followed by a translation. The translation component is itself a rigid motion, hence conformal, so we need only check to see if the linear component is conformal. Further, we can assume our matrix \( M \) is written with respect to the so-called standard basis, that is, the orthonormal basis usually denoted by the vectors \( \mathbf{i} \) and \( \mathbf{j} \), so the first column of \( M \) is \( \mathbf{i}' = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and the second column of \( M \) is \( \mathbf{j}' = M \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). If \( \mathbf{i}' \) and \( \mathbf{j}' \) are also orthogonal with \( |\mathbf{i}'| = |\mathbf{j}'| \) then the transformation will preserve the absolute measure of angles between any pair of vectors \( \mathbf{v}, \mathbf{w} \).

**Exercise 36.** Remind yourself why this last statement is true.  
**Suggestion:** Show that

\[
\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{\mathbf{v}' \cdot \mathbf{w}'}{|\mathbf{v}'| |\mathbf{w}'|}
\]

**Exercise 37.** Show that if \( T \) is a similarity then the matrix part \( M \) of its representation is either

\[
\begin{pmatrix} k \cos \theta & -k \sin \theta \\
 k \sin \theta & k \cos \theta \end{pmatrix}
\]

or

\[
\begin{pmatrix} k \cos \theta & -k \sin \theta \\
 k \sin \theta & -k \cos \theta \end{pmatrix}
\]

for some choice of \( \theta \) and some \( k > 0 \). It follows that \( T \) is an isometry if \( k = 1 \), in which case we have

\[
M^t = M^{-1}
\]

where \( M^t \) is the transpose of \( M \).

Apparently the matrix \( M \) of an isometry has determinant \( \pm 1 \). (However, there are collineations with \( \det M = \pm 1 \) that are not isometries.) First consider the determinant 1 case

\[
M = \begin{pmatrix} \cos \theta & -\sin \theta \\
 \sin \theta & \cos \theta \end{pmatrix}
\]
$M$ rotates any point $P = (x, y)$ about the origin $O$ through the angle $\theta$. We have

$$T(x, y) = M \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} p + x \cos \theta - y \sin \theta \\ q + y \cos \theta + x \sin \theta \end{array} \right)$$

Now, a rotation followed by a translation preserves orientation, so $T$ is either a rotation or translation. If $\theta = 0$ then $T$ is the translation $(x, y) \mapsto (x + p, y + q)$.

**Exercise 38.** Explain why $T$ is a translation for no other value of $\theta \in [0, 2\pi)$.

Thus, if $\theta \neq 0$ then $T$ is a rotation.

*Where is the center $Q$ of the rotation?*

To find $Q$ we express $T$ as a sequence of reflections. If we choose any two lines $l_1, l_2$ through $O$ such that the angle from $l_1$ to $l_2$ is $\frac{\theta}{2}$, then the matrix part of $T$ is $l_2 \circ l_1$. Let $l_1$ be the line through $O$ and $(p, q)$ and let $l_2$ be the rotation of $l_1$ about $O$ through $\frac{\theta}{2}$. Now let $l_3$ be the line through $O$ perpendicular to $l_1$, and let $l_4$ be the perpendicular bisector of the segment between $O$ and $(p, q)$. Then $l_4 \circ l_3$ is the translation part of $T$, so

$$T = l_4 \circ (l_3 \circ l_2 \circ l_1)$$

where we have associated the three lines through $O$. Recall (Theorem 5.2) that

$$l_3 \circ l_2 \circ l_1 = l$$

where $l$ is obtained by rotating $l_1$ about $O$ through the angle from $l_2$ to $l_3$. Now $l = l_3$ only if $\theta = 0$. Otherwise $l$ intersects $l_4$ at the center $Q$ of the rotation $T = l_4 \circ l$.

**Exercise 39.** Show that the angle from $l$ to $l_4$ is $\frac{\theta}{2}$.

**Exercise 40.** Show that the coordinates of $Q$ are

$$\left( \frac{1}{2} \left( p - q \cot \frac{\theta}{2} \right), \frac{1}{2} \left( q + p \cot \frac{\theta}{2} \right) \right)$$

Where is $Q$ if $T$ is a half-turn? Where is $Q$ if $\theta = \frac{\pi}{2}$.

Of course, if $\theta \neq 0$ the coordinates of $Q$ are the unique solution $(x, y)$ of the system

$$p + x \cos \theta - y \sin \theta = x$$
$$q + y \cos \theta + x \sin \theta = y$$

because a rotation has a unique fixed point.
Exercise 41. Suppose $k > 0, k \neq 1$. Show that

$$T(x, y) = \begin{pmatrix} k \cos \theta & -k \sin \theta \\ k \sin \theta & k \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$$

is a dilation. Find the center $C$ of the dilation. Show that $C$ is on the circle

$$\left(x + \frac{p}{k^2 - 1}\right)^2 + \left(y + \frac{q}{k^2 - 1}\right)^2 = k^2 \frac{p^2 + q^2}{(k^2 - 1)^2}$$

Note. Given $P = (p, q)$, if $C$ is on the circle given by $k$ then

$$PC \over OC = k$$

When $k = 1$ we obtain the perpendicular bisector of $OP$, which is a member of this Apollonian family of inversive lines determined by $O$ and $P$. (Apollonius of Perga, about a century after Euclid; see Brannan/Esplen/Gray, p. 317.)

Exercise 42. Choose a point $P = (p, q)$. Sketch the ‘circles of Apollonius’ for several values of $k$, including $k = 1$.

Next consider the determinant $-1$ case

$$M = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$M$ reflects any point $P = (x, y)$ in a line through the origin.

Exercise 43. Show that the line of reflection is

$$y = x \tan \frac{\theta}{2}$$

What is the line of reflection if $\theta = \pi$?

We have

$$T(x, y) = M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p + x \cos \theta + y \sin \theta \\ q + x \sin \theta - y \cos \theta \end{pmatrix}$$

Now, $T$ is either a reflection or glide because $T$ reverses orientation ($M$ reverses orientation and the translation part preserves orientation). We can express $T$ as a sequence of three reflections. Let $l_1$ be the line of reflection for $M$ (see
Exercise 43. Let \( l_2 \) be the line through \( O \) perpendicular to the line through \( O \) and \((p, q)\) and let \( l_3 \) be the perpendicular bisector of the segment between \( O \) and \((p, q)\). Then
\[
T = l_3 \circ l_2 \circ l_1
\]

**Case 1.** If the line through \( O \) and \((p, q)\) is perpendicular to the line of reflection for \( M \) then \( l_2 = l_1 \), so
\[
T = l_3
\]
that is, \( T \) is the reflection in the perpendicular bisector of the segment between \( O \) and \((p, q)\).

Exercise 44. If \( T = l_3 \) show that
\[
T(x, y) = \begin{pmatrix}
\frac{q^2 - p^2}{p^2 + q^2} & -\frac{2pq}{p^2 + q^2} \\
-\frac{2pq}{p^2 + q^2} & \frac{p^2 + q^2}{p^2 + q^2}
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
p \\
q
\end{pmatrix}
\]

**Case 2.** Otherwise, \( l_1 \) is a transversal to the parallel lines \( l_2 \) and \( l_3 \), so \( T \) is a glide (Theorem 5.3). Note that if \((p, q)\) is on \( l_1 \) then \( l_1 \) is the axis of the glide and the translation component is the vector from \( O \) to \((p, q)\).

Exercise 45. If \((p, q)\) is on \( l_1 \) show that
\[
T(x, y) = \begin{pmatrix}
\frac{p^2 - q^2}{p^2 + q^2} & \frac{2pq}{p^2 + q^2} \\
\frac{2pq}{p^2 + q^2} & \frac{q^2 - p^2}{p^2 + q^2}
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
p \\
q
\end{pmatrix}
\]

In all other cases, then, \( T \) is a glide.

*Where is the axis of the glide?*

Since \( l_1 \) is a transversal to the parallel lines \( l_2 \) and \( l_3 \) we apply Theorem 5.3 with \( l_1 = m \). In this theorem \( m \) is the third reflection whereas \( l_1 \) is our first reflection. However, \( T^{-1} \) has the same axis as \( T \) and
\[
T^{-1} = l_1 \circ l_2 \circ l_3
\]
and so we apply Theorem 5.3 with
\[
l_2 = \ell \\
l_3 = k
\]
Exercise 46. Show that the axis of $T$ is the line
\[(2x - p) \sin \theta = (2y - q) (\cos \theta + 1)\]

Exercise 47. Show that the translation component of $T$ after reflection in this axis is
\[(I + M) \begin{pmatrix} \frac{1}{2}p \\ \frac{1}{2}q \end{pmatrix}\]
where $I$ is the identity matrix.

Exercise 48. Let $T(x, y) = kM \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$ with $k > 0$ and $k \neq 1$. Show that $T$ has a unique fixed point $P$ and that $P$ is on the circle
\[
\left( x + \frac{p}{k^2 - 1} \right)^2 + \left( y + \frac{q}{k^2 - 1} \right)^2 = k^2 \left( p^2 + q^2 \right) \frac{1}{(k^2 - 1)^2}
\]
In this case, $T$ is an orientation-reversing dilation.
Week 10. Transformation of General Figures

The transformation of points and lines by collineations is relatively straightforward. How would we find the image of an arbitrary figure without transforming each point individually? Most figures of interest satisfy a particular Cartesian equation when we represent them in the coordinate plane, and we can use the Fundamental Transformation Principle (which is easy to prove):

**Theorem 10.1 (FTP)** Let $F$ be a figure described in the coordinate plane by the equation $f(x, y) = 0$ and let $T$ be a transformation of the plane. Then the figure $T(F)$ is described by the equation $f(T^{-1}(x, y)) = 0$

**Proof.** If $P$ is a point on $T(F)$ then $T^{-1}(P)$ is a point on $F$. Let $P = (x, y)$. Then $T^{-1}(x, y)$ must satisfy $f = 0$. ■

**Exercise 49.** Let

$$T(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}, \quad ad - bc \neq 0$$

and let $l$ be the line $Ax + By = C$. Use the FTP to show that $T(l)$ is the line

$$(dA - cB)x + (aB - bA)y = (dp - bq)A + (aq - cp)B + (ad - bc)C$$

Of course, to find $T(l)$ we could compute the images of of any two points on $l$ and find the line through them, but most figures are not so easily determined. Here is an important example that demonstrates the necessity of defining similarity by transformations.

**Theorem 10.2** Any two parabolas are similar.

Let’s prove this first without using the FTP. We know that any parabola $F$ has a vertex and an axis of symmetry. The axis of symmetry is the line $l$ through the vertex $V$ such that $l(F) = F$. Given two parabolas $F$ and $F'$, we need to find a similarity transformation $T$ such that

$$T(F) = F'$$

Begin by applying the translation that takes $V$ to $V'$. The translation takes $l$, the axis of $F$, to a line $m$ through $V'$ so we can now apply a rotation about $V'$ that takes $m$ to $l'$, the axis of $F'$. After these two rigid motions the parabolas are as in the following figure, with common vertex and common axis. Since we can always reduce the situation to this configuration, let’s call the common vertex $O$.
Now we want a transformation of the plane that maps one parabola onto the other. For this we recall the focus/directrix characterization of a parabola $F$ as the locus of points $P$ such that the distance between $P$ and a given point $F$ is equal to the distance between $P$ and a given line $d$. Choose any point $P$ on $F$ and let $Q$ be the foot of the perpendicular through $P$ on $d$. Then $PF = PQ$. 
Let $F'$ be the focus of the parabola $\mathcal{F}'$ and let

$$k = \frac{OF'}{OF}$$

The dilation about $O$ with factor $k$ maps $F$ to $F'$. Let $F'P'Q'$ be the dilation of triangle $FPQ$. Then $P'F' = P'Q'$ and, since dilation maps any line to a parallel line, $Q'$ is the foot of the perpendicular through $P'$ on $\overline{d'}$.

Thus $P'$ is on $\mathcal{F}'$, and we have proved Theorem 10.2. ■

Note that the dilation about $O$ with factor $\frac{1}{k}$ maps $\mathcal{F}'$ to $\mathcal{F}$.

Now let’s use the FTP to prove this theorem. In the coordinate plane we can assume (after translation and rotation, if necessary) that $\mathcal{F}$ is in standard position, that is, the parabola is described by the equation $y = ax^2$ for some $a > 0$. The dilation about the origin $O$ with factor $k$ is

$$T(x, y) = \left( \begin{array}{cc} k & 0 \\ 0 & k \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} kx \\ ky \end{array} \right)$$

whereby

$$T^{-1}(x, y) = \left( \begin{array}{c} \frac{1}{k}x \\ \frac{1}{k}y \end{array} \right)$$

Therefore, $T(\mathcal{F})$ is described by the equation

$$\frac{1}{k}y = a \left( \frac{1}{k}x \right)^2$$

$$y = \frac{a}{k}x^2$$

Since we can also assume that $\mathcal{F}'$ is in standard position, this parabola is described by $y = a'x^2$ for some $a' > 0$. We choose $k$ so that

$$\frac{a}{k} = a'$$

$$k = \frac{a}{a'}$$

Exercise 50. Determine the dilation that maps the parabola $4y = x^2$ to the parabola $8y = x^2$. Where is the focus and directrix for each parabola?
Suggested Exercises for Study

- Express the collineation in Exercise 32 in analytic form \( T(x, y) \). Find the center of this dilation and the factor \( k \).

- Find the center and factor \( k \) of the dilation

\[
T(x, y) = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

- Let

\[
T(x, y) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

For what value(s) of \( \theta \) is \( T \) a translation?

- For what value(s) of \( \theta \) does

\[
T(x, y) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

represent a glide? \textbf{Note:} A glide does not fix any point.

- Let

\[
T(x, y) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \sqrt{3} \\ \frac{1}{2} \sqrt{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}
\]

Find \( p \) and \( q \) so that \( T \) is a reflection.

- Use the \textbf{FTP} to show that \( T_1 \) and \( T_2 \) map the parabola \( y = x^2 \) to the same parabola, where

\[
T_1(x, y) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}
\]

\[
T_2(x, y) = \begin{pmatrix} -9 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}
\]

Is either collineation a similarity transformation?

- Use the \textbf{FTP} to find a collineation that maps the unit circle \( x^2 + y^2 = 1 \) to the ellipse

\[
4x^2 + 9y^2 = 1
\]
For any $k > 0$, let $T$ be the dilation

$$T(x, y) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Sketch the hyperbola $\mathcal{F}$ described by

$$x^2 - y^2 = 1$$

and use the FTP to find $T(\mathcal{F})$. Then sketch $T(\mathcal{F})$ for several values of $k$. 
Appendix. Invariant line of a glide

Find the axis of the glide $T = c \circ b \circ a$ for

1) Interpret $T$ either as reflection in $a$ followed by a rotation about $A$, or as a rotation about $C$ followed by reflection in $c$. For example, choose $T = (c \circ b) \circ a$.

2) Rotate $b$ and $c$ about $A$ until one of them is parallel to $a$. If $b \parallel a$ then $c$ is now a transversal.
3) Let $K$ be the foot of the perpendicular to $a$ through $A$.

4) Let $a'$ be the line through $K$ parallel to the transversal $c$, let $b'$ be the line through $K$ perpendicular to $a'$, and let $c'$ be the line through $A$ parallel to $b'$.
Then $T = c' \circ b' \circ a'$. The axis of this glide $a'$ is uniquely determined, though $b'$ and $c'$ can be any two lines of their parallel pencil separated by the same distance.

Exercise. In the original factorization $T = c \circ b \circ a$, the inverse of $T$ is $T^{-1} = a \circ b \circ c$, which has the same axis as $T$. Find the axis of the glide $b \circ a \circ c$ and
its inverse $c \circ a \circ b$, and the axis of the glide $a \circ c \circ b$ and its inverse $b \circ c \circ a$. Sketch all three axes together with the triangle $ABC$. Let $\alpha, \beta, \gamma$ be the angle measures at $A, B, C$ and suppose each is less than $\frac{\pi}{2}$. Show that the three glide axes meet at angles that measure $\pi - 2\alpha$, $\pi - 2\beta$, and $\pi - 2\gamma$. (First show that the axis through the foot of each perpendicular makes the same angle with that side as the angle opposite that side.) What happens if $\alpha = \frac{\pi}{2}$? What happens if $\alpha > \frac{\pi}{2}$?