DEFINING MULTIPLICATION VIA THE LEVER

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The objects in our system are line segments and point masses. We will use “m” and “d” to denote points and line segments respectively. We will use the ordered four tuple \((m_1, d_\alpha, m_2, d_\beta)\) as a short-hand notation to denote the system depicted below.

\[
\begin{array}{c}
d_\alpha \\
\hline \\
\hline \\
\hline  \\
\hline \\
d_\beta \\
\end{array}
\]

Note each \((m_1, d_\alpha, m_2, d_\beta)\) either balances the lever or causes the lever to move. If the lever balances we write \((m_1, d_\alpha) = (m_2, d_\beta)\)

Axioms:
(i) Given \((m_1, d_\alpha, m_2)\) there exists a unique \(d_\beta\) such that \((m_1, d_\alpha) = (m_2, d_\beta)\). Moreover, we will write \(f((m_1, d_\alpha, m_2)) = d_\beta\)
(ii) \((m_1, d_\alpha, m_1, d_\alpha \mp \epsilon) \cup (m_1, d_\beta, m_1, d_\beta \pm \epsilon)\) balances the lever.
(iii) If \((m_1, d_\alpha) = (m_2, d_\beta)\) and \((m_1, d_\sigma) = (m_4, d_\tau)\) then \((m_1, d_\alpha, m_2, d_\beta) \cup (m_3, d_\gamma, m_4, d_\sigma)\) balances the lever.
(iv) If \(n\) is a whole number then \((m_1, nd_\alpha) = (nm_1, d_\alpha)\).

**Definition 1.** Let \(m_u\) and \(d_u\) be an arbitrarily chosen point mass and line segment respectively. We will call \(m_u\) and \(d_u\) our unit mass and unit length respectively.

**Definition 2.** If \((m_1, d_\alpha) = (m_u, d_1)\) and \((m_2, d_u) = (m_u, d_2)\) then we define \(d_1 L d_2 := f(m_2, d_1, m_u)\). We read “\(d_1 L d_2\)” as “\(d_1\) lever multiplied by \(d_2\)”.

**Theorem 1.** \((m_u, d_1) = (m_\beta, d_2)\) if and only if \(d_1 L d_\alpha = d_2 L d_\beta\).

**Proof.** Observe that by definition of multiplication we have \((m_u, d_1) = (m_u, d_1 L d_\alpha)\) and \((m_\beta, d_2) = (m_u, d_2 L d_\beta)\). Claim now follows from this observation.

**Theorem 2.** \(d_1 L (d_2 + d_3) = d_1 L d_2 + d_1 L d_3\)

**Proof.** By definition we have \((m_2, d_\alpha) = (m_u, d_2)\) and \((m_3, d_\alpha) = (m_u, d_3)\). By axiom (iii), \((m_2, d_u, m_u, d_2) \cup (m_3, d_u, m_u, d_3)\) balances the lever. Moreover, from axiom (ii) with \(\epsilon = d_2\) we have \((m_u, d_2, m_u, d_2 - d_2) \cup (m_u, d_3, m_u, d_3 + d_2)\) balances the lever. From the foregoing statements we can now deduce that \((m_2 + m_3, d_\alpha) = (m_u, d_2 + d_3)\) which in turn implies that \(d_1 L (d_2 + d_3) = f(m_2 + m_3, d_1, m_u)\). Claim now follows by noting that \(d_1 L (d_2 + d_3) = f(m_2 + m_3, d_1, m_u) = f(m_2, d_1, m_u) + f(m_3, d_1, m_u) = d_1 L d_2 + d_1 L d_3\).

We next show that lever multiplication with whole numbers reduces to repeated addition.
Lemma 1. If $a, b$ are whole numbers then $(ab)aL(ba) = (ab)d_u = (bd_u)L(ad_u)$.

Proof. By axiom (iv) we have, $(am_u, d_u) = (m_u, ad_u)$, $(bm_u, a) = (m_u, bd_u)$, $(am_u, bd_u) = (m_u, b)u$, and $(bm_u, ad_u) = (m_u, (ab)d_u)$. By definition of multiplication this implies that $(ab)d_uL(bd_u) = f(bm_u, ad_u, m_u) = (ab)d_u$ and $(bd_u)L(ad_u) = f(am_u, bd_u, m_u) = (ba)d_u$. Claim now follows since $(ab)d_u = (ba)d_u$. □

Lemma 2. If $a, b \neq 0$, $c$, and $d \neq 0$ are whole numbers then $(\frac{a}{b}d_u)L(\frac{c}{d}u) = \frac{ac}{bd}d_u = (\frac{c}{d}u)L(\frac{a}{b}d_u)$.

Proof. By axiom (iv) we have, $(\frac{a}{b}m_u, d_u) = (am_u, \frac{1}{b}d_u)$ and $(am_u, \frac{1}{b}d_u) = (m_u, \frac{a}{b}d_u)$. This implies that $(\frac{a}{b}m_u, d_u) = (m_u, \frac{a}{b}d_u)$. Similarly, it can be shown that $(\frac{c}{d}m_u, d_u) = (m_u, \frac{c}{d}d_u)$. By axiom (iv), $(\frac{a}{b}m_u, \frac{c}{d}d_u)) = (am_u, \frac{cd}{bd}d_u)$ and $(am_u, \frac{cd}{bd}d_u) = (m_u, \frac{cd}{bd}d_u)$. This implies that $(\frac{a}{b}m_u, \frac{c}{d}d_u) = (m_u, \frac{cd}{bd}d_u)$. In a similar fashion it can be shown that $(\frac{c}{d}m_u, \frac{a}{b}d_u) = (m_u, \frac{cd}{bd}d_u)$. Now, $\frac{ac}{bd}d_u = \frac{cd}{bd}d_u$. By definition of lever multiplication claim now follows. □

Suppose $a$ and $b$ are natural numbers with $a < b$. In the figure below, note that when $(0, d_u), (d_1, 0)$ is parallel to $(0, \frac{ad_u}{b}), (y, 0)$ we have $x = \frac{d_1}{b}$ and $y = \frac{ad_1}{b}$. If $a > b$ we get a similar conclusion.

Next that by Axiom (iv) we have, $(\frac{am_u}{b}, d_1) = (am_u, \frac{d_1}{b}) = (m_u, \frac{ad_1}{b})$. This implies that $d_1L(\frac{ad_u}{b}) = \frac{ad_1}{b}$. The following Lemma summarizes our observations.

Lemma 3. $d_1L(\frac{ad_u}{b}) = \frac{ad_1}{b}$. Moreover, $(0, \frac{ad_u}{b}), (d_1L(\frac{ad_u}{b}), 0)$ is parallel to $(0, d_u), (d_1, 0)$.

Theorem 3. $(d_1, 0), (0, d_u)$ is parallel to $(d_1Ld_2, 0), (0, d_2)$.

Proof. Suppose that $(d_1, 0), (0, d_u)$ is not parallel to $(d_1Ld_2, 0), (0, d_2)$. Let $(x, 0)$ be the unique point such that $(d_1, 0), (0, d_u)$ is parallel to $(x, 0), (0, d_2)$. Suppose that $x < d_1Ld_2$. There exists a unique $\tilde{d}_2$ such that $(\frac{x(d_1Ld_2)}{d_2}, 0), (0, d_2)$ is parallel to $(d_1, 0), (0, d_u)$. By design $d_2 < \tilde{d}_2$. By the density of the rational numbers there exists a rational number $\frac{a}{b}$ such that $d_2 < \frac{ad_1}{b} < \tilde{d}_2$. By previous lemma this implies $d_1L(\frac{ad_u}{b}) = \frac{ad_1}{b} < d_1Ld_2$. This is a contradiction since $d_2 < \frac{ad_1}{b}$. The case when $x > d_1Ld_2$ is handled in a similar way. Hence it must be the case that $(d_1, 0), (0, d_u)$ is parallel to $(d_1Ld_2, 0), (0, d_2)$.

The following Lemma is Proposition 37 in book one of Euclid's Elements.
Lemma 4. \( A(\triangle ABC) = A(\triangle ABC_1) \) if and only if \( CC_1 \parallel AB \)

Theorem 4. Let \( \triangle ABC \) and \( \triangle AB_1C_1 \) be two right triangles with \( \angle BAC = \angle BAC_1 = 90 \). The areas of the two triangles are equal if and only if \( BC_1 \) is parallel to \( B_1C \).

Proof. Refer to figure below. Note \( A(\triangle ABC) = A(\triangle AB_1C_1) \iff A(\triangle BC_1C) = A(\triangle B_1BC_1) \). Also by previous lemma, \( A(\triangle BC_1C) = A(\triangle B_1BC_1) \iff BC_1 \parallel B_1C \). Hence claim follows.

Theorem 5. \( d_1Ld_2 = d_3Ld_4 \) if and only if area of the rectangle determined by \( d_1 \) and \( d_2 \) equals the area of rectangle determined by \( d_3 \) and \( d_4 \).

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