WHEN DOES A CROSS PRODUCT ON $\mathbf{R}^n$ EXIST?

PETER F. MCLOUGHLIN

It is probably safe to say that just about everyone reading this article is familiar with the cross product and the dot product. However, what many readers may not be aware of is that the familiar properties of the cross product in three space can only be extended to $\mathbf{R}^7$. Students are usually first exposed to the cross and dot products in a multivariable calculus or linear algebra course. Let $u \neq 0$, $v \neq 0$, $\widetilde{v}$, and $\widetilde{w}$ be vectors in $\mathbf{R}^3$ and let a, b, c, and d be real numbers. For review, here are some of the basic properties of the dot and cross products:

(i) $\frac{(u \cdot v)}{\sqrt{(u \cdot u)(v \cdot v)}} = cos\theta$ (where θ is the angle formed by the vectors)
(ii) $\frac{||(u \times v)||}{\sqrt{(u \cdot u)(v \cdot v)}} = sin\theta$
(iii) $u \cdot (u \times v) = 0$ and $v \cdot (u \times v) = 0.$ (perpendicular property)
(iv) $(u \times v) \cdot (u \times v) + (u \cdot v)^2 = (u \cdot u)(v \cdot v)$ (Pythagorean property)
(v) $((au + b\widetilde{u}) \times (cv + d\widetilde{v})) = ac(u \times v) + ad(u \times \widetilde{v}) + bc(\widetilde{u} \times v) + bd(\widetilde{u} \times \widetilde{v})$

We will refer to property (v) as the bilinear property. The reader should note that properties (i) and (ii) imply the Pythagorean property. Recall, if A is a square matrix then $|A|$ denotes the determinant of A. If we let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ then we have:

$$u \cdot v = x_1y_1 + x_2y_2 + x_3y_3 \qquad and \qquad (u \times v) = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

It should be clear that the dot product can easily be extended to $\mathbf{R}^n$, however, it is not so obvious how the cross product could be extended. In general, we define a cross product on $\mathbf{R}^n$ to be a function from $\mathbf{R}^n \times \mathbf{R}^n$ (here "$\times$" denotes the Cartesian product) to $\mathbf{R}^n$ that satisfies the perpendicular, Pythagorean, and bilinear properties. A natural question to ask might be: "Can we extend the cross product to $\mathbf{R}^n$ for $n > 3$ and if so how?". If we only required the cross product to have the perpendicular and bilinear properties then there are many ways we could define the product. For example if $u = (x_1, x_2, x_3, x_4)$ and $v = (y_1, y_2, y_3, y_4)$ we could define $u \times v := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1, 0)$. As the reader can check, this definition can be shown to satisfy the perpendicular and bilinear properties and it can be easily extended to $\mathbf{R}^n$. However, the Pythagorean property does not always hold for this product. For example, if we take $u = (0, 0, 0, 1)$ and $v = (1, 0, 0, 0)$ then, as the reader can check, the Pythagorean property fails. Another way we might try to extend the cross product is using the determinant. A natural way to do this on $\mathbf{R}^4$ would be to consider the following determinant:

$$\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}$$

As the reader can verify, this determinant idea can be easily extended to $\mathbf{R}^n$. Recall that if two rows of a square matrix repeat then the determinant is zero. This implies, for i=1,2 or 3, that:

$$\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} \cdot (a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3 + a_{i4}e_4) = \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} = 0$$

It follows our determinant product has the perpendicular property on its row vectors. Note, however, for $n > 3$ the determinant product acts on more than two vectors which implies it cannot be a candidate for a cross product on $\mathbf{R}^n$.
Surprisingly, a cross product can exist in $\mathbf{R}^n$ if and only if n is 0, 1, 3 or 7. The intention of this article is to provide a new constructive elementary proof (i.e. could be included in an linear algebra undergraduate text) of this classical result which is accessible to a wide audience. The proof provided in this paper holds for any finite-dimensional inner product space over $\mathbf{R}$. It has recently been brought to my attention that the approach taken in this paper is similar to the approach taken in an article in The American Mathematical Monthly written back in 1967 (see [13]).

In 1943 Beno Eckmann, using algebraic topology, gave the first proof of this result (he actually proved the result under the weaker condition that the cross product is only continuous (see [1] and [11])). The result was later extended to nondegenrate symmetric bilinear forms over fields of characteristic not equal to two (see [2], [3], [4], [5], and [6]). It turns out that there are intimate relationships between the cross product, quaternions and octonions, and Hurwitz' theorem (also called the "1,2,4 8 Theorem" named after Adolf Hurwitz, who proved it in 1898). Throughout the years many authors have written on the history of these topics and their intimate relationships to each other (see [5], [6],[7], [8], and [9]).

1. BASIC IDEA OF PROOF

From this point on, the symbol "$\times$" will always denote cross product. Moreover, unless otherwise stated, u_i will be understood to mean a unit vector. In $\mathbf{R}^3$ the relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_1 \times e_3 = -e_2$ determine the right-hand-rule cross product. The crux of our proof will be to show that in general if a cross product exists on $\mathbf{R}^n$ then we can always find an orthonormal basis $e_1, e_2 \ldots e_n$ where for any $i \neq j$ there exists a k such that $e_i \times e_j = ae_k$ with $a = 1$ or -1. How could we generate such a set? Before answering this question, we will need some basic properties of cross products. These properties are contained in the following Lemma and Corollary:

Lemma 1. *Suppose u, v and w are vectors in $\boldsymbol{R}^n$. If a cross product exists on $\boldsymbol{R}^n$ then it must have the following properties:*
(1.1) $w \cdot (u \times v) = -u \cdot (w \times v)$
(1.2) $u \times v = -v \times u$ *which implies* $u \times u = 0$
(1.3) $v \times (v \times u) = (v \cdot u)v - (v \cdot v)u$
(1.4) $w \times (v \times u) = -((w \times v) \times u) + (u \cdot v)w + (w \cdot v)u - 2(w \cdot u)v$

For a proof of this Lemma see [12]. Recall that two nonzero vectors u and v are orthogonal if and only if $u \cdot v = 0$.

Corollary 1. *Suppose u, v and w are orthogonal unit vectors in $\boldsymbol{R}^n$. If a cross product exists on $\boldsymbol{R}^n$ then it must have the following properties:*
(1.5) $u \times (u \times v) = -v$
(1.6) $w \times (v \times u) = -((w \times v) \times u)$

Proof. Follows from previous Lemma by substituting $(u \cdot v) = (u \cdot w) = (w \cdot v) = 0$ and $(w \cdot w) = (v \cdot v) = (u \cdot u) = 1$. ☐

Now that we have our basic properties of cross products, let's start to answer our question. Observe that $\{e_1, e_2, e_3\} = \{e_1, e_2, e_1 \times e_2\}$. Let's see how we could generalize and/or extend this idea.

Recall u_i always denotes a unit vector, the symbol "$\perp$" stands for orthogonal, and the symbol "$\times$" denotes cross product. We will now recursively define a sequence of sets $\{S_k\}$ by $S_0 := \{u_0\}$ and $S_k := S_{k-1} \cup \{u_k\} \cup (S_{k-1} \times u_k)$ where $u_k \perp S_{k-1}$.
Let's explicitly compute S_0, S_1, S_2, and S_3.
$S_0 = \{u_0\}$; $S_1 = S_0 \cup \{u_1\} \cup (S_0 \times u_1) = \{u_0, u_1, u_0 \times u_1\}$; $S_2 = S_1 \cup \{u_2\} \cup (S_1 \times u_2) = \{u_0, u_1, u_0 \times u_1, u_2, u_0 \times u_2, u_1 \times u_2, (u_0 \times u_1) \times u_2\}$; $S_3 = S_2 \cup \{u_3\} \cup (S_2 \times u_3) = \{u_0, u_1, u_0 \times u_1, u_2, u_0 \times u_2, u_1 \times u_2, (u_0 \times u_1) \times u_2, u_3, u_0 \times u_3, u_1 \times u_3, (u_0 \times u_1) \times u_3, u_2 \times u_3, (u_0 \times u_2) \times u_3, (u_1 \times u_2) \times u_3, ((u_0 \times u_1) \times u_2) \times u_3\}$

Note: S_1 corresponds to $\{e_1, e_2, e_1 \times e_2\}$.
We define $S_i \times S_j := \{u \times v \ \ \textit{where} \ \ u \in S_i \ \ \textit{and} \ \ v \in S_j\}$.
We also define $\pm S_i := S_i \cup (-S_i)$.
In the following two Lemmas we will show that S_n is an orthonormal set which is closed under the cross product. This in turn implies that a cross can only exist in $\mathbf{R}^n$ if $n = 0$ or $n = |S_k|$.

Lemma 2. $S_1 = \{u_0, u_1, u_1 \times u_0\}$ *is an orthonormal set. Moreover,* $S_1 \times S_1 = \pm S_1$.

Proof. By definition of cross product we have:
(1) $u_0 \cdot (u_0 \times u_1) = 0$
(2) $u_1 \cdot (u_0 \times u_1) = 0$
It follows S_1 is an orthogonal set.
Next by (1.1), (1.2), and (1.3) we have:
(1) $u_1 \times (u_0 \times u_1) = u_0$
(2) $u_0 \times (u_0 \times u_1) = -u_1$

(3) $(u_0 \times u_1) \cdot (u_0 \times u_1) = u_0 \cdot (u_1 \times (u_0 \times u_1)) = u_0 \cdot u_0 = 1$
It follows S_1 is an orthonormal set and $S_1 \times S_1 = \pm S_1$. □

Lemma 3. *S_k is an orthonormal set. Moreover, $S_k \times S_k = \pm S_k$ and $|S_k| = 2^{k+1} - 1$.*

Proof. We proceed by induction. When $k = 1$ claim follows from previous Lemma. Suppose S_{k-1} is an orthonormal set, $S_{k-1} \times S_{k-1} = \pm S_{k-1}$, and $|S_{k-1}| = 2^k - 1$. Let $y_1, y_2 \in S_{k-1}$. By definition any element of S_k is of the form y_1, u_k, or $y_1 \times u_k$. By (1.1), (1.2), and (1.6) we have:
(1) $y_1 \cdot (y_2 \times u_k) = u_k \cdot (y_1 \times y_2) = 0$ since $u_k \perp S_{k-1}$ and $y_1 \times y_2 \in S_{k-1}$.
(2) $(y_1 \times u_k) \cdot (y_2 \times u_k) = u_k \cdot ((u_k \times y_1) \times y_2) = -u_k \cdot (u_k \times (y_1 \times y_2)) = 0$
It follows S_k is an orthogonal set.
Next by (1.1), (1.2), (1.5), and (1.6) we have:
(1) $(y_1 \times u_k) \times (y_2 \times u_k) = -y_1 \times (u_k \times (y_2 \times u_k)) = -y_1 \times y_2$
(2) $y_1 \times (y_2 \times u_k) = -((y_1 \times y_2) \times u_k)$ and $y_1 \times (y_1 \times u_k) = -u_k$
(3) $(y_1 \times u_k) \cdot (y_1 \times u_k) = y_1 \cdot (u_k \times (y_1 \times u_k)) = y_1 \cdot y_1 = 1$
It follows S_k is an orthonormal set and $S_k \times S_k = \pm S_k$. Lastly, note $|S_k| = 2\,|S_{k-1}| + 1 = 2^{k+1} - 1$ □

Lemma 3 tells us how to construct a multiplication table for the cross product on S_k. Let's take a look at the multiplication table in $\mathbf{R}^3$. In order to simplify the notation for the basis elements we will make the following assignments: $e_1 := u_0$, $e_2 := u_1$, $e_3 := u_0 \times u_1$

$\times$	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

In the table above we used (1.2), (1.5) and (1.6) to compute each product. In particular, $e_1 \times e_3 = u_0 \times (u_0 \times u_1) = -u_1 = -e_2$. The other products are computed in a similar way.
Now when $v = x_1e_1 + x_2e_2 + x_3e_3$ and $w = y_1e_1 + y_2e_2 + y_3e_3$ we have $v \times w = (x_1e_1 + x_2e_2 + x_3e_3) \times (y_1e_1 + y_2e_2 + y_3e_3) = x_1y_1(e_1 \times e_1) + x_1y_2(e_1 \times e_2) + x_1y_3(e_1 \times e_3) + x_2y_1(e_2 \times e_1) + x_2y_2(e_2 \times e_2) + x_2y_3(e_2 \times e_3) + x_3y_1(e_3 \times e_1) + x_3y_2(e_3 \times e_2) + x_3y_3(e_3 \times e_3)$. By the multiplication table above we can simplify this expression. Hence,
$v \times w = (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3$
The reader should note that this cross product is the standard right-hand-rule cross product.

Let's next take a look at the multiplication table in $\mathbf{R}^7$. In order to simplify the notation for the basis elements we will make the following assignments: $e_1 := u_0$, $e_2 := u_1$, $e_3 := u_0 \times u_1$, $e_4 := u_2$, $e_5 := u_0 \times u_2$, $e_6 := u_1 \times u_2$, $e_7 := (u_0 \times u_1) \times u_2$.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

In the table above we used (1.2), (1.5) and (1.6) to compute each product. In particular, $e_5 \times e_6 = (u_0 \times u_2) \times (u_1 \times u_2) = -(u_0 \times u_2) \times (u_2 \times u_1) = ((u_0 \times u_2) \times u_2) \times u_1 = -(u_2 \times (u_0 \times u_2)) \times u_1 = -u_0 \times u_1 = -e_3$. The other products are computed in a similar way.

Let's look at $v \times w$ when $v = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and $w = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$. Using the bilinear property of the cross product and the multiplication table for $\mathbf{R}^7$ we have:

$v \times w = (-x_3y_2 + x_2y_3 - x_5y_4 + x_4y_5 - x_6y_7 + x_7y_6)e_1 + (-x_1y_3 + x_3y_1 - x_6y_4 + x_4y_6 - x_7y_5 + x_5y_7)e_2 + (-x_2y_1 + x_1y_2 - x_7y_4 + x_4y_7 - x_5y_6 + x_6y_5)e_3 + (-x_1y_5 + x_5y_1 - x_2y_6 + x_6y_2 - x_3y_7 + x_7y_3)e_4 + (-x_4y_1 + x_1y_4 - x_2y_7 + x_7y_2 - x_6y_3 + x_3y_6)e_5 + (-x_7y_1 + x_1y_7 - x_4y_2 + x_2y_4 - x_3y_5 + x_5y_3)e_6 + (-x_5y_2 + x_2y_5 - x_4y_3 + x_3y_4 - x_1y_6 + x_6y_1)e_7$

Lemma 4. *If $u = (u_0 \times u_1) + (u_1 \times u_3)$ and $v = (u_1 \times u_2) - (((u_0 \times u_1) \times u_2) \times u_3)$ then $u \times v = 0$ and $u \perp v$.*

Proof. Using (1.2), (1.5), (1.6), and the bilinear property it can be shown that $u \times v = 0$. Next, note that $u_0 \times u_1$, $u_1 \times u_3$, $u_1 \times u_2$, and $((u_0 \times u_1) \times u_2) \times u_3$ are all elements of S_i when $i \geq 3$. By Lemma 3 all these vectors form an orthonormal set. This implies $u \perp v$. □

Lemma 5. *If $u = (u_0 \times u_1) + (u_1 \times u_3)$ and $v = (u_1 \times u_2) - (((u_0 \times u_1) \times u_2) \times u_3)$ then $(u \cdot u)(v \cdot v) \neq (u \times v) \cdot (u \times v) + (u \cdot v)^2$*

Proof. Using the previous Lemma and Lemma 3, we have $(u \times v) = 0$, $(u \cdot u) = 2$, $(v \cdot v) = 2$, and $(u \cdot v) = 0$. Hence, claim follows. □

Theorem 1. *A cross product can exist in $\boldsymbol{R}^n$ if and only if n=0, 1, 3 or 7. Moreover, there exists orthonormal bases S_1 and S_2 for $\boldsymbol{R}^3$ and $\boldsymbol{R}^7$ respectively such that $S_i \times S_i = \pm S_i$ for $i = 1$ or 2.*

Proof. By Lemma 3 we have that a cross product can only exist in $\mathbf{R}^n$ if $n = 2^{k+1} - 1$. Moreover, Lemmas 4 and 5 tells us that if we try to define a cross product in $\mathbf{R}^{2^{k+1}-1}$ then the Pythagorean property does not always hold when $k \geq 3$. It follows $\mathbf{R}^0$, $\mathbf{R}^1$, $\mathbf{R}^3$, and $\mathbf{R}^7$ are the only spaces on which a cross product can exist. It is left as an exercise to show that the cross product that we defined above for $\mathbf{R}^7$ satisfies all the properties of a cross product and the zero map defines a valid cross product on $\mathbf{R}^0$, and $\mathbf{R}^1$. Lastly, Lemma 3 tells us how to generate orthonormal bases S_1 and S_2 for $\mathbf{R}^3$ and $\mathbf{R}^7$ respectively such that $S_i \times S_i = \pm S_i$ for $i = 1$ or 2. □

ACKNOWLEDGMENTS

I would like to thank Dr. Daniel Shapiro, Dr. Alberto Elduque, and Dr. John Baez for their kindness and encouragement and for reading over and commenting on various drafts of this paper. I would also like to thank the editor and all the anonymous referees for reading over and commenting on various versions of this paper. This in turn has lead to overall improvements in the mathematical integrity, readability, and presentation of the paper.

References

[1] B. Eckmann, "Stetige Losungen linearer Gleichungssysteme", Comment. Math. Helv. 15 (1943), 318-339.

[2] R.B. Brown and A. Gray, "Vector cross products", Comment. Math. Helv. 42 (1967), 222-236.

[3] M. Rost, "On the dimension of a composition algebra", Doc. Math. 1 (1996), No. 10, 209-214 (electronic).

[4] K. Meyberg, "Trace formulas in vector product algebras", Commun. Algebra 30 (2002), no. 6, 2933-2940.

[5] Alberto Elduque, "Vector Cross Products", Talk presented at the Seminario Rubio de Francia of the Universidad de Zaragoza on April 1, 2004. Electronic copy found at: http://www.unizar.es/matematicas/algebra/elduque/Talks/crossproducts.pdf

[6] W.S. Massey, "Cross Products of Vectors in Higher Dimensional Euclidean Spaces", The American Mathematical Monthly (1983), Vol. 90, No. 10, 697-701.

[7] John, Baez, "The Octonions", Bull. Amer. Math. Soc. 39 (02): 145205. Electronic copy found at: http://www.ams.org/journals/bull/2002-39-02/S0273-0979-01-00934-X/S0273-0979-01-00934-X.pdf

[8] D. B. Shapiro, *Compositions of Quadratic Forms*, W. de Gruyter Verlag, 2000.

[9] Bruce W. Westbury, "Hurwitz' Theorem", 2009. Electronic copy found at: http://arxiv.org/abs/1011.6197

[10] A. Hurwitz, (1898). "Ueber die Composition der quadratischen Formen von beliebig vielen Variabeln (On the composition of quadratic forms of arbitrary many variables)" (in German). Nachr. Ges. Wiss. Gttingen: 309-316.

[11] Beno Eckmann, "Continuous solutions of linear equations An old problem, its history, and its solution", Exposition Math, 9 (1991) pp.

[12] Peter F. Mcloughlin, "Basic Properties of Cross Products", Electronic copy found at: http://www.math.csusb.edu/faculty/pmclough/BP.pdf

[13] Bertram Walsh, "The Scarcity of Cross Products on Euclidean Spaces", The American Mathematical Monthly (Feb. 1967), Vol. 74, No. 2, 188-194.

E-mail address: `pmclough@csusb.edu`