WHEN DOES A CROSS PRODUCT ON \( \mathbb{R}^n \) EXIST?

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It is probably safe to say that just about everyone reading this article is familiar with the cross product and the dot product. However, what many readers may not be aware of is that the familiar properties of the cross product in three space can only be extended to \( \mathbb{R}^7 \). Students are usually first exposed to the cross and dot products in a multivariable calculus or linear algebra course. Let \( u \neq 0, v \neq 0, \tilde{v}, \) and \( \tilde{w} \) be vectors in \( \mathbb{R}^3 \) and let \( a, b, c, \) and \( d \) be real numbers. For review, here are some of the basic properties of the dot and cross products:

(i) \( \frac{(u \cdot v)}{\sqrt{(u \cdot u)(v \cdot v)}} = \cos \theta \) (where \( \theta \) is the angle formed by the vectors)

(ii) \( \frac{|u \times v|}{\sqrt{(u \cdot u)(v \cdot v)}} = \sin \theta \)

(iii) \( u \cdot (u \times v) = 0 \) and \( v \cdot (u \times v) = 0. \) (perpendicular property)

(iv) \( (u \times v) \cdot (u \times v) + (u \cdot v)^2 = (u \cdot u)(v \cdot v) \) (Pythagorean property)

(v) \( ((au + b\tilde{u}) \times (cv + d\tilde{v})) = ac(u \times v) + ad(u \times \tilde{v}) + bc(\tilde{u} \times v) + bd(\tilde{u} \times \tilde{v}) \)

We will refer to property (v) as the bilinear property. The reader should note that properties (i) and (ii) imply the Pythagorean property. Recall, if \( A \) is a square matrix then \( |A| \) denotes the determinant of \( A \). If we let \( u = (x_1, x_2, x_3) \) and \( v = (y_1, y_2, y_3) \) then we have:

\[
\begin{align*}
u \cdot v &= x_1 y_1 + x_2 y_2 + x_3 y_3 \quad \text{and} \quad (u \times v) = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
\end{align*}
\]

It should be clear that the dot product can easily be extended to \( \mathbb{R}^n \), however, it is not so obvious how the cross product could be extended. In general, we define a cross product on \( \mathbb{R}^n \) to be a function from \( \mathbb{R}^n \times \mathbb{R}^n \) (here “\( \times \)” denotes the Cartesian product) to \( \mathbb{R}^n \) that satisfies the perpendicular, Pythagorean, and bilinear properties. A natural question to ask might be: “Can we extend the cross product to \( \mathbb{R}^n \) for \( n > 3 \) and if so how?” If we only required the cross product to have the perpendicular and bilinear properties then there are many ways we could define the product. For example if \( u = (x_1, x_2, x_3, x_4) \) and \( v = (y_1, y_2, y_3, y_4) \) we could define \( u \times v := (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1, 0) \). As the reader can check, this definition can be shown to satisfy the perpendicular and bilinear properties and it can be easily extended to \( \mathbb{R}^n \). However, the Pythagorean property does not always hold for this product. For example, if we take \( u = (0, 0, 0, 1) \) and \( v = (1, 0, 0, 0) \) then, as the reader can check, the Pythagorean property fails. Another way we might try to extend the cross product is using the determinant. A
natural way to do this on $\mathbb{R}^4$ would be to consider the following determinant:

$$
\begin{vmatrix}
    e_1 & e_2 & e_3 & e_4 \\
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34}
\end{vmatrix}
$$

As the reader can verify, this determinant idea can be easily extended to $\mathbb{R}^n$. Recall that if two rows of a square matrix repeat then the determinant is zero. This implies, for $i=1,2$ or $3$, that:

$$
\begin{vmatrix}
    e_1 & e_2 & e_3 & e_4 \\
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34}
\end{vmatrix}
\cdot
(a_i e_1 + a_i e_2 + a_i e_3 + a_i e_4) =
\begin{vmatrix}
    a_{i1} & a_{i2} & a_{i3} & a_{i4} \\
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34}
\end{vmatrix}
= 0
$$

It follows our determinant product has the perpendicular property on its row vectors. Note, however, for $n > 3$ the determinant product acts on more than two vectors which implies it cannot be a candidate for a cross product on $\mathbb{R}^n$.

Surprisingly, a cross product can exist in $\mathbb{R}^n$ if and only if $n$ is 0, 1, 3 or 7. The intention of this article is to provide a new constructive elementary proof (i.e. could be included in an linear algebra undergraduate text) of this classical result which is accessible to a wide audience. The proof provided in this paper holds for any finite-dimensional inner product space over $\mathbb{R}$. It has recently been brought to my attention that the approach taken in this paper is similar to the approach taken in an article in The American Mathematical Monthly written back in 1967 (see [13]).

In 1943 Beno Eckmann, using algebraic topology, gave the first proof of this result (he actually proved the result under the weaker condition that the cross product is only continuous (see [1] and [11])). The result was later extended to nondegenerate symmetric bilinear forms over fields of characteristic not equal to two (see [2], [3], [4], [5], and [6]). It turns out that there are intimate relationships between the cross product, quaternions and octonions, and Hurwitz’ theorem (also called the “1,2,4, 8 Theorem” named after Adolf Hurwitz, who proved it in 1898). Throughout the years many authors have written on the history of these topics and their intimate relationships to each other (see [5], [6],[7], [8], and [9]).

1. BASIC IDEA OF PROOF

From this point on, the symbol “×” will always denote cross product. Moreover, unless otherwise stated, $u_i$ will be understood to mean a unit vector. In $\mathbb{R}^3$ the relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_1 \times e_3 = -e_2$ determine the right-hand-rule cross product. The crux of our proof will be to show that in general if a cross product exists on $\mathbb{R}^n$ then we can always find an orthonormal basis $e_1, e_2, \ldots, e_n$ where for any $i \neq j$ there exists a $k$ such that $e_i \times e_j = a e_k$ with $a = 1$ or $-1$. How could we generate such a set? Before answering this question, we will need some basic properties of cross products. These properties are contained in the following Lemma and Corollary:
Lemma 1. Suppose \( u, v \) and \( w \) are vectors in \( \mathbb{R}^3 \). If a cross product exists on \( \mathbb{R}^n \) then it must have the following properties:

\[
\begin{align*}
(1.1) \quad w \cdot (u \times v) &= -u \cdot (w \times v) \\
(1.2) \quad u \times v &= -v \times u \quad \text{which implies} \quad u \times u = 0 \\
(1.3) \quad v \times (u \times u) &= (v \cdot u)u - (v \cdot u)u \\
(1.4) \quad w \times (v \times u) &= -(w \times v)u + (u \cdot w)v + (v \cdot w)u - 2(u \cdot v)u
\end{align*}
\]

For a proof of this Lemma see [12]. Recall that two nonzero vectors \( u \) and \( v \) are orthogonal if and only if \( u \cdot v = 0 \).

Corollary 1. Suppose \( u, v \) and \( w \) are orthogonal unit vectors in \( \mathbb{R}^n \). If a cross product exists on \( \mathbb{R}^n \) then it must have the following properties:

\[
\begin{align*}
(1.5) \quad u \times (u \times v) &= -v \\
(1.6) \quad w \times (v \times u) &= -(w \times v)u
\end{align*}
\]

Proof. Follows from previous Lemma by substituting \( (u \cdot v) = (u \cdot w) = (v \cdot v) = 0 \) and \( (v \cdot w) = (v \cdot v) = (u \cdot u) = 1 \).

Now that we have our basic properties of cross products, let’s start to answer our question. Observe that \( \{e_1, e_2, e_3\} = \{e_1, e_2, e_1 \times e_2\} \). Let’s see how we could generalize and/or extend this idea.

Recall \( u_i \) always denotes a unit vector, the symbol “\( \perp \)” stands for orthogonal, and the symbol “\( \times \)” denotes cross product. We will now recursively define a sequence of sets \( S_k \) by \( S_0 := \{u_0\} \) and \( S_k := S_{k-1} \cup \{u_k\} \cup (S_{k-1} \times u_k) \) where \( u_k \perp S_{k-1} \).

Let’s explicitly compute \( S_0, S_1, S_2, \) and \( S_3 \).

\[
S_0 = \{u_0\}; \quad S_1 = S_0 \cup \{u_1\} \cup (S_0 \times u_1) = \{u_0, u_1, u_0 \times u_1\}; \\
S_2 = S_1 \cup \{u_2\} \cup (S_1 \times u_2) = \{u_0, u_1, u_0 \times u_1, u_2, u_0 \times u_2, u_1 \times u_2, (u_0 \times u_1) \times u_2\}; \\
S_3 = S_2 \cup \{u_3\} \cup (S_2 \times u_3) = \{u_0, u_1, u_0 \times u_1, u_2, u_0 \times u_2, u_1 \times u_2, (u_0 \times u_1) \times u_2, u_3, u_0 \times u_3, u_1 \times u_3, (u_0 \times u_1) \times u_3, (u_0 \times u_2) \times u_3, (u_1 \times u_2) \times u_3\}
\]

Note: \( S_1 \) corresponds to \( \{e_1, e_2, e_1 \times e_2\} \).

We define \( S_1 \times S_j := \{u \times v \text{ where } u \in S_i \text{ and } v \in S_j\} \).

We also define \( \pm S_i := S_i \cup (-S_i) \).

In the following two Lemmas we will show that \( S_n \) is an orthonormal set which is closed under the cross product. This in turn implies that a cross can only exist in \( \mathbb{R}^n \) if \( n = 0 \) or \( n = |S_k| \).

Lemma 2. \( S_1 = \{u_0, u_1, u_1 \times u_0\} \) is an orthonormal set. Moreover, \( S_1 \times S_1 = \pm S_1 \).

Proof. By definition of cross product we have:

\[
\begin{align*}
(1) \quad u_0 \cdot (u_0 \times u_1) &= 0 \\
(2) \quad u_1 \cdot (u_0 \times u_1) &= 0
\end{align*}
\]

It follows \( S_1 \) is an orthogonal set.

Next by (1.1), (1.2), and (1.3) we have:

\[
\begin{align*}
(1) \quad u_1 \times (u_0 \times u_1) &= u_0 \\
(2) \quad u_0 \times (u_0 \times u_1) &= -u_1
\end{align*}
\]
(3) \((u_0 \times u_1) \cdot (u_0 \times u_1) = u_0 \cdot (u_1 \times (u_0 \times u_1)) = u_0 \cdot u_0 = 1\)

It follows that \(S_1\) is an orthonormal set and \(S_1 \times S_1 = \pm S_1\) \(\square\)

**Lemma 3.** \(S_k\) is an orthonormal set. Moreover, \(S_k \times S_k = \pm S_k\) and \(|S_k| = 2^{k+1} - 1\)

**Proof.** We proceed by induction. When \(k = 1\) claim follows from previous Lemma. Suppose \(S_{k-1}\) is an orthonormal set, \(S_{k-1} \times S_{k-1} = \pm S_{k-1}\), and \(|S_{k-1}| = 2^k - 1\).

Let \(y_1, y_2 \in S_{k-1}\). By definition any element of \(S_k\) is of the form \(y_1, u_k\), or \(y_1 \times u_k\).

By (1.1), (1.2), and (1.6) we have:

1. \((y_1 \times u_k) \cdot (y_1 \times u_k) = u_k \cdot (y_1 \times y_2) = 0\) since \(u_k \perp S_{k-1}\) and \(y_1 \times y_2 \in S_{k-1}\).
2. \((y_1 \times u_k) \cdot (y_2 \times u_k) = u_k \cdot ((u_k \times y_1) \times y_2) = 0\) since \(u_k \perp S_{k-1}\).

It follows that \(S_k\) is an orthonormal set.

Next by (1.1), (1.2), (1.5), and (1.6) we have:

1. \(y_1 \times (y_2 \times u_k) = -(y_1 \times u_k) \times (y_2 \times u_k) = 0\)
2. \(y_1 \times (y_2 \times u_k) = -(y_1 \times y_2) \times u_k\) and \(y_1 \times (u_k \times u_k) = 0\)
3. \((y_1 \times u_k) \cdot (y_1 \times u_k) = y_1 \cdot (u_k \times (y_1 \times u_k)) = y_1 \cdot y_1 = 1\)

It follows that \(S_k\) is an orthonormal set and \(S_k \times S_k = \pm S_k\). Lastly, note that \(|S_k| = 2|S_{k-1}| + 1 = 2^{k+1} - 1\) \(\square\)

Lemma 3 tells us how to construct a multiplication table for the cross product on \(S_k\). Let’s take a look at the multiplication table in \(\mathbb{R}^3\). In order to simplify the notation for the basis elements we will make the following assignments: \(e_1 := u_0, e_2 := u_1, e_3 := u_0 \times u_1\)

<table>
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<th>(e_1)</th>
<th>(e_2)</th>
<th>(e_3)</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>(e_3)</td>
<td>(-e_2)</td>
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<tr>
<td>(e_2)</td>
<td>(-e_3)</td>
<td>0</td>
<td>(e_1)</td>
</tr>
<tr>
<td>(e_3)</td>
<td>(e_2)</td>
<td>(-e_1)</td>
<td>0</td>
</tr>
</tbody>
</table>

In the table above we used (1.2), (1.5), and (1.6) to compute each product. In particular, \(e_1 \times e_3 = u_0 \times (u_0 \times u_1) = -u_1 = -e_2\). The other products are computed in a similar way.

Now when \(v = x_1 e_1 + x_2 e_2 + x_3 e_3\) and \(w = y_1 e_1 + y_2 e_2 + y_3 e_3\) we have \(v \times w = (x_1 e_1 + x_2 e_2 + x_3 e_3) \times (y_1 e_1 + y_2 e_2 + y_3 e_3) = x_1 y_1 (e_1 \times e_1) + x_1 y_2 (e_1 \times e_2) + x_1 y_3 (e_1 \times e_3) + x_2 y_1 (e_2 \times e_1) + x_2 y_2 (e_2 \times e_2) + x_2 y_3 (e_2 \times e_3) + x_3 y_1 (e_3 \times e_1) + x_3 y_2 (e_3 \times e_2) + x_3 y_3 (e_3 \times e_3)\).

By the multiplication table above we can simplify this expression. Hence,

\(v \times w = (x_2 y_1 - x_3 y_2) e_1 + (x_3 y_1 - x_1 y_3) e_2 + (x_1 y_2 - x_2 y_1) e_3\)

The reader should note that this cross product is the standard right-hand-rule cross product.

Let’s next take a look at the multiplication table in \(\mathbb{R}^7\). In order to simplify the notation for the basis elements we will make the following assignments: \(e_1 := u_0, e_2 := u_1, e_3 := u_0 \times u_1, e_4 := u_2, e_5 := u_0 \times u_2, e_6 := u_1 \times u_2, e_7 := (u_0 \times u_1) \times u_2\).
In the table above we used (1.2), (1.5) and (1.6) to compute each product. In particular, \( e_5 \times e_6 = (u_0 \times u_2) \times (u_1 \times u_2) = -(u_0 \times u_2) \times (u_2 \times u_1) = (u_0 \times u_2) \times u_1 = -(u_2 \times (u_0 \times u_2)) \times u_1 = -u_0 \times u_1 = -e_3 \). The other products are computed in a similar way.

Let’s look at \( v \times w \) when \( v = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \) and \( w = (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \). Using the bilinear property of the cross product and the multiplication table for \( \mathbb{R}^7 \) we have:

\[
v \times w = (-x_3 y_2 + x_2 y_3 - x_5 y_1 + x_4 y_5 - x_6 y_7 + x_7 y_6) e_1 + (-x_1 y_3 + x_3 y_1 - x_6 y_4 + x_4 y_6 - x_7 y_5 + x_5 y_7) e_2 + (-x_2 y_1 + x_1 y_2 - x_7 y_4 + x_4 y_7 - x_5 y_6 + x_6 y_5) e_3 + (-x_1 y_5 + x_5 y_1 - x_2 y_6 + x_6 y_2 - x_3 y_7 + x_7 y_3) e_4 + (-x_4 y_1 + x_1 y_4 - x_2 y_7 + x_7 y_2 - x_6 y_3 + x_3 y_6) e_5 + (-x_7 y_1 + x_1 y_7 - x_4 y_2 + x_2 y_4 - x_3 y_5 + x_5 y_3) e_6 + (-x_5 y_2 + x_2 y_5 - x_4 y_3 + x_3 y_4 - x_1 y_6 + x_6 y_1) e_7\]

**Lemma 4.** If \( u = (u_0 \times u_1) + (u_1 \times u_3) \) and \( v = (u_1 \times u_2) - (u_0 \times u_1) \times u_2 \times u_3 \) then \( u \times v = 0 \) and \( u \perp v \).

**Proof.** Using (1.2), (1.5), (1.6), and the bilinear property it can be shown that \( u \times v = 0 \). Next, note that \( u_0 \times u_1, u_1 \times u_3, u_1 \times u_2, \) and \((u_0 \times u_1) \times u_2 \times u_3\) are all elements of \( S_i \) when \( i \geq 3 \). By Lemma 3 all these vectors form an orthonormal set. This implies \( u \perp v \). \( \square \)

**Lemma 5.** If \( u = (u_0 \times u_1) + (u_1 \times u_3) \) and \( v = (u_1 \times u_2) - ((u_0 \times u_1) \times u_2) \times u_3 \) then \((u \cdot u)(v \cdot v) = (u \cdot v) \cdot (u \cdot v) + (u \cdot v)^2\)

**Proof.** Using the previous Lemma and Lemma 3, we have \((u \cdot v) = 0, (u \cdot u) = 2, (v \cdot v) = 2, \) and \((u \cdot v) = 0\). Hence, claim follows. \( \square \)

**Theorem 1.** A cross product can exist in \( \mathbb{R}^n \) if and only if \( n=0, 1, 3 \) or 7. Moreover, there exists orthonormal bases \( S_1 \) and \( S_2 \) for \( \mathbb{R}^3 \) and \( \mathbb{R}^7 \) respectively such that \( S_i \times S_i = \pm S_i \) for \( i = 1 \) or 2.

**Proof.** By Lemma 3 we have that a cross product can only exist in \( \mathbb{R}^n \) if \( n = 2^k + 1 \). Moreover, Lemmas 4 and 5 tells us that if we try to define a cross product in \( \mathbb{R}^{2^k+1} \) then the Pythagorean property does not always hold when \( k \geq 3 \). It follows \( \mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^3, \) and \( \mathbb{R}^7 \) are the only spaces on which a cross product can exist. It is left as an exercise to show that the cross product that we defined above for \( \mathbb{R}^7 \) satisfies all the properties of a cross product and the zero map defines a valid cross product on \( \mathbb{R}^0 \), and \( \mathbb{R}^1 \). Lastly, Lemma 3 tells us how to generate orthonormal bases \( S_1 \) and \( S_2 \) for \( \mathbb{R}^3 \) and \( \mathbb{R}^7 \) respectively such that \( S_i \times S_i = \pm S_i \) for \( i = 1 \) or 2. \( \square \)
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