5.5 \[ H_n \]

\[ 0) \quad \left( \binom{n'}{0} \right) + \binom{n'+1}{1} + \binom{n'+2}{2} + \ldots + \binom{n'+r'}{r'} = \binom{n'+r+1}{r'} \]

\[ 1) \quad \binom{m}{1} \binom{n}{r+1} + \binom{m}{2} \binom{n}{r+2} + \ldots + \binom{m}{m} \binom{n}{r+m} = \binom{m+n}{m+r} \]

Show \( 1) \Rightarrow 0) \)

\[ m+r = r' \]
\[ m+n = n'+r'+1 \]

\[ \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r} \]

\[ \sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r} \]

\[ \frac{m!}{k!(m-k)!} \cdot \frac{n!}{(n-r-k)!(r+k)!} \]
Ex. Calculate the coefficient of $x^{16}$ in $(x^2 + x^3 + x^4 + \ldots)^5$.

$$(x^2 + x^3 + x^4 + \ldots)^5 = \left[\frac{x^2}{1-x}\right]^5$$

$$x^{10} \cdot (1 + x + x^2 + \ldots)^5$$

The coefficient of $x^{16}$ is the same as the coefficient of $x^6$ in $(1 + x + x^2 + \ldots)^5$.

$$\left(1 + x + x^2 + \ldots\right)^5 = \frac{1}{(1-x)^5}$$

5. The coefficient of $x^n$ in \(\frac{1}{(1-x)^n}\)

is \(\left(\begin{array}{c} r+n-1 \\ r \end{array}\right) = \binom{n+5-1}{6} = 210\).
Ex. Use generating functions to find the number of ways to collect $15 from 20 distinct people if each of the first 19 people can give $0 or $1 and the twentieth person can give $0, $1, or $5.

\[ g(x) = (1 + x)^{19} (1 + x + x^5) \]

We need to find the coefficient of $x^{15}$. This equals the sum of the coefficients of $x^{15}$, $x^{14}$, and $x^{10}$ in $(1 + x)^{19}$.

\[ 3 \cdot (1 + x)^{19} = 1 + \binom{19}{1} x + \binom{19}{2} x^2 + \ldots + \binom{19}{19} x^{19} \]

\[ \binom{19}{15} + \binom{19}{14} + \binom{19}{10} = 107,882 \]
Ex. How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

\[ g(x) = \left( x^2 + x^3 + x^4 + x^5 + x^6 \right)^7 \]

Need the coefficient of \( x^{25} \).

\[ = \left[ x^2 \left( 1 + x + x^2 + x^3 + x^4 \right) \right]^7 \]

\[ = x^{14} \left( 1 + x + x^2 + x^3 + x^4 \right)^7 \]

\[ \uparrow \text{need } x^{11} \]

By 1, \( \left( \frac{1 - x^5}{1 - x} \right)^7 \)

\[ = (1 - x^5)^7 \cdot \frac{1}{(1 - x)^7} \]

By 4 and 5,

\[ = \left( 1 - \frac{x^5}{1} + \frac{x^{10}}{2} x - \frac{x^{15}}{3} x^3 + \ldots \right) \cdot \left( 1 + \frac{7}{1} x + \ldots \right) \]

\[ \text{need } r = 11, \quad r = 6, \quad r = 1 \]
we get \(1 \cdot \binom{17}{11} - \binom{7}{1} \cdot \binom{12}{6} + \binom{7}{2} \cdot \binom{12}{1}
\]

\[
= \boxed{6,055}
\]