EXERCISES (page 19):

Problems:

23. If $R$ is a field, prove that the map $R \rightarrow \text{Frac}(R)$, given by $a \mapsto \frac{a}{1}$, is an isomorphism.

Conversely, if $R$ is a domain and the map $a \mapsto \frac{a}{1}$ is an isomorphism, then $R$ is a field.

24. If $\phi : R \rightarrow S$ is an isomorphism between domains, prove that there is an isomorphism $\text{Frac}(R) \rightarrow \text{Frac}(S)$, namely, $\frac{a}{b} \mapsto \frac{\phi(a)}{\phi(b)}$.

25. Let $R$ be a subring of a field $F$, and let $K$ be the intersection of all the subfields of $F$ that contain $R$. Prove that $K \cong \text{Frac}(R)$.

26. (i) If $\phi : R \rightarrow S$ is an isomorphism, then $\phi^{-1} : S \rightarrow R$ is also an isomorphism.

(ii) If $\varphi : R \rightarrow S$ and $\psi : S \rightarrow T$ are ring homomorphisms, then so is $\psi \varphi : R \rightarrow T$.

27. If $a \in R$ is a unit in $R$ and if $\phi : R \rightarrow S$ is a ring map, then $\phi(a)$ is a unit in $S$.

28. (i) If $R$ is a ring, prove that $\sigma : \mathbb{Z}[x] \rightarrow R$, where $\sigma(f(x)) \mapsto c_0$, the constant term of $f(x)$, is a ring map.

(ii) What is $\ker \phi$?

29. (i) If $R$ is a ring map, prove that $\sigma^* : \mathbb{Z}[x] \rightarrow S[x]$, defined by $\Sigma r_i x^i \mapsto \Sigma \sigma(r_i) x^i$, is also a ring map.

(ii) If $\tau : S \rightarrow T$ is a ring map, prove that $(\tau \sigma)^* : \mathbb{Z}[x] \rightarrow T[x]$ is equal to $\tau^* \sigma^*$.

(iii) Prove that if $\sigma$ is an isomorphism, then so is $\sigma^*$.

30. (i) The intersection of any family of ideals in $R$ is an ideal in $R$. Conclude that if $X$ is any subset of a ring $R$, there is a smallest ideal, denoted by $(X)$, containing $X$. One calls $(X)$ the ideal generated by $X$, namely, the intersection of all the ideals in $R$ that contain $X$.

(ii) Prove that $(X)$ is the "smallest" ideal containing $X$ in the following sense: $(X)$ is an ideal containing $X$ and if $J$ is any ideal in $R$ containing $X$, then $(X) \subseteq J$.

31. (i) If $a \in R$, prove $\{ra : r \in R\}$ is an ideal generated by $a$; it is called the principal ideal generated by $a$, and it is denoted by $(a)$.

(ii) If $a_1, a_2, \ldots, a_n$ are elements in a ring, prove that the set of all linear combinations, $I = \{r_1a_1 + r_2a_2 + \ldots + r_na_n : r_i \in R, i = 1, 2, \ldots, n\}$, is equal to $(a_1, \ldots, a_n)$, the ideal generated by $\{a_1, a_2, \ldots, a_n\}$.

34. (i) The set $I$ of all $f(x) \in \mathbb{Z}[x]$ having even constant term is an ideal in $\mathbb{Z}[x]$; it consists of all linear combinations of $x$ and $2$; that is, $I = (2, x)$.

(ii) Prove that $(2, x)$ is not a principal ideal in $\mathbb{Z}[x]$.

35. Prove that if $F$ is a field and $S$ is a ring, then a ring map $\varphi : F \rightarrow S$ must be an injection and $\text{im } \varphi$ is a subfield of $S$ isomorphic to $F$. 
EXERCISES (page 23):

Problems:

38. Prove the correspondence Theorem for rings. If $I$ is a proper ideal in a ring $R$, then there is a bijection from the family of all intermediate ideals $J$, where $I \subset J \subset R$, to the family of all ideals in $R/I$, given by $J \mapsto \pi(J) = J/I = \{a + I : a \in J\}$, where $\pi : R \longrightarrow R/I$ is the natural map. Moreover, if $J \subset J'$ are intermediate ideals, then $\pi(J) \subset \pi(J')$.

EXERCISES (page 30):

Problems:

43. In the ring $R = \mathbb{Z}[x]$, show that $x$ and 2 are relatively prime, but there are no polynomials $f(x)$ and $g(x) \in \mathbb{Z}[x]$ with $1 = xf(x) + 2g(x)$.

44. Let $f(x) = \Pi(x - a_i) \in F[x]$, where $F$ is a field and $a_i \in F$ for all $i$. Show that $f(x)$ has no repeated roots [i.e., $f(x)$ is not a multiple of $(x - a)^2$ for any $a \in F$] if and only if $(f(x), f'(x)) = 1$, where $f'(x)$ is the derivative of $f(x)$.

46. Prove that $\mathbb{Z}_2[x]/I$ is a field, where $p(x) = x^3 + x + 1 \in \mathbb{Z}_2[x]$ and $I = (p(x))$.

47. If $R$ is a ring and $a \in R$, let $e_a : R[x] \longrightarrow R$ be evaluation at $a$. Prove that $\text{Ker} \ e_a$ consists of all the polynomials over $R$ having $a$ as a root, and so $\text{Ker} \ e_a = (x - a)$, the principal ideal generated by $x - a$.

48. Let $F$ be a field, and let $f(x), g(x) \in F[x]$. Prove that if $\partial(f) \leq \partial(g) = n$ and if $f(a) = g(a)$ for $n + 1$ elements $a \in F$, then $f(x) = g(x)$