Exercises 2.4 (page 72): Problems:

3. Find all the subsequential limits of the following sequences.
   b. \( \{ \cos \frac{4k\pi}{2} \} \), \( \{ \cos \frac{(4k+1)\pi}{2} \} = \{0\} \), and \( \{ \cos \frac{(4k+2)\pi}{2} \} = \{-1\} \). Thus the set of subsequential limits is \( \{0, 1, -1\} \).
   f. \( \{ (1.5 + (-1)^n)^n \} \).
      \( \{(1.5 + (-1)^{2k-1})^{2k-1}\} = (\frac{3}{2})^{2k-1} \) cgs to 0
      and \( \{(1.5 + (-1)^{2k})^{2k}\} = (2.5)^{2k} \) dgs to \( \infty \). Thus the set of subsequential limits is \( \{0, \infty\} \).

7. Determine the limit points and the isolated points of each of the following sets.
   c. \( (0, 1) \cup \{2\} \)
   The set of limit points is \([0, 1]\) and 2 is the only isolated point of \((0, 1) \cup \{2\}\).
   e. \( \mathbb{R} \setminus \mathbb{Q} \)
   Let \( p \in \mathbb{R} \). Show that \( p \) is a limit point of \( \mathbb{R} \setminus \mathbb{Q} \).
   Let \( \epsilon > 0 \). Then since \( p < p + \epsilon, 3\epsilon \in \mathbb{R} \setminus \mathbb{Q} \) such that \( p < r < p + \epsilon \). Thus \( N_r(p) = (p - \epsilon, p + \epsilon) \) contains a point of \( \mathbb{R} \setminus \mathbb{Q} \) other than \( p \). \( \implies p \) is a limit point of \( \mathbb{R} \setminus \mathbb{Q} \). Hence the set of limit points of \( \mathbb{R} \setminus \mathbb{Q} \) is \( \mathbb{R} \). Thus \( \mathbb{R} \setminus \mathbb{Q} \) has no isolated points.

Exercises 2.6 (page 85): Problems:

1. If \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences in \( \mathbb{R} \), prove (without using Theorem 2.6.4) that \( \{a_n + b_n\} \) and \( \{a_nb_n\} \) are Cauchy.

   Let \( \epsilon > 0 \). Find \( n_0 \in \mathbb{N} \) such that if \( m, n \geq n_0 \) then \(|a_n + b_n - (a_m + b_m)| < \epsilon\).

   \( \{a_n\} \) is Cauchy \( \implies \forall n_1 \in \mathbb{N} \) such that \(|a_n - a_m| < \frac{\epsilon}{2} \forall m, n \geq n_1 \) and \( \{b_n\} \) is Cauchy \( \implies \exists n_2 \in \mathbb{N} \) such that \(|b_n - b_m| < \frac{\epsilon}{2} \forall m, n \geq n_2 \). Choose \( n_0 = \max\{n_1, n_2\} \). If \( m, n \geq n_0 \) then \(|a_n + b_n - (a_m + b_m)| = |a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\).

   Next, \( \{a_nb_n\} \) is Cauchy.

   Let \( \epsilon > 0 \). Find \( n_0 \in \mathbb{N} \) such that if \( m, n \geq n_0 \) then \(|a_nb_n - (a_m b_m)| < \epsilon\).

   \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences \( \implies \) both are bounded \( \implies \exists M_1 > 0 \) and \( M_2 > 0 \) such that \(|a_n| < M_2 \forall n \in \mathbb{N} \) and \( |b_n| < M_1 \forall n \in \mathbb{N} \). Also \( \{a_n\} \) is Cauchy \( \implies \exists n_1 \in \mathbb{N} \) such that \(|a_n - a_m| < \frac{\epsilon}{2M_1} \forall m, n \geq n_1 \) and \( \{b_n\} \) is Cauchy \( \implies \exists n_2 \in \mathbb{N} \) such that \(|b_n - b_m| < \frac{\epsilon}{2M_2} \forall m, n \geq n_2 \). Choose \( n_0 = \max\{n_1, n_2\} \). If \( m, n \geq n_0 \) then \(|a_nb_n - (a_m b_m)| = |(a_n b_n) - (a_m b_m) + (a_m b_n) - (a_m b_m)| = |a_n(b_n - b_m) + b_n(a_m - a_n)| \leq |a_n||b_n - b_m| + |b_n||a_m - a_n| < M_2(\frac{\epsilon}{2M_1}) + M_1(\frac{\epsilon}{2M_2}) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\).

2. Use only the definition to show that the following sequences are or are not Cauchy.
   a. \( \{\frac{2^n - 1}{2^n}\} \)
   We prove that \( \{\frac{2^n - 1}{2^n}\} \) is Cauchy.
   Let \( \epsilon > 0 \). Find \( n_0 \in \mathbb{N} \) such that if \( m, n \geq n_0 \) then \(|\frac{2^n - 1}{2^n} - \frac{2^m - 1}{2^m}| < \epsilon\). Now \(|\frac{2^n - 1}{2^n} - \frac{2^m - 1}{2^m}| = \frac{2^n - 2^m}{2^n 2^m} = \frac{2^n - 2^m}{2^{n+m}} = \frac{1}{2^m} < \frac{\epsilon}{2^n}\)
Use the definition to prove that the sequence \( \left\{ \frac{2^m(2^n-1)-2^n(2^m-1)}{2^{m+n}} \right\} \) converges, where \( m, n \geq 0 \).

\[
\frac{|2^m(2^n-1)-2^n(2^m-1)|}{2^{m+n}} = \frac{|2^m-2^n|}{2^{m+n}} \quad (\text{assume } m > n < \frac{2^m}{2^n}).
\]

\[
\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}. \quad \text{Choose } n_0 > \frac{1}{\epsilon}. \quad \text{Then if } m > n \geq n_0 \text{ then } \left| \frac{2^m-2^n}{2^{m+n}} \right| < \frac{1}{n} < \frac{1}{\epsilon} = \epsilon.
\]

(b) \( \left\{ \frac{2n^2+1}{n^2} \right\} \)
We prove that \( \left\{ \frac{2n^2+1}{n^2} \right\} \) is Cauchy.

Let \( \epsilon > 0 \). Find \( n_0 \in \mathbb{N} \) such that if \( m, n \geq n_0 \) then \( \left| \frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2} \right| < \epsilon \). Now \( \left| \frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2} \right| = \frac{m^2-n^2}{m^2+n^2} \) (assume \( m > n \)).

\[
\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}. \quad \text{Choose } n_0 > \frac{1}{\epsilon}. \quad \text{Then if } m > n \geq n_0 \text{ then } \left| \frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2} \right| < \frac{1}{n} < \frac{1}{\epsilon} = \epsilon.
\]

3. Use the definition to prove that the sequence \( \left\{ \frac{1}{n^2} \right\} \) is Cauchy.
Let \( \epsilon > 0 \). Find \( n_0 \in \mathbb{N} \) such that if \( m, n \geq n_0 \) then \( \left| \frac{1}{n^2} - \frac{1}{m^2} \right| < \epsilon \). Now \( \left| \frac{1}{n^2} - \frac{1}{m^2} \right| = \frac{m^2-n^2}{m^2+n^2} \) (assume \( m > n \)).

\[
\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}. \quad \text{Choose } n_0 > \frac{1}{\epsilon}. \quad \text{Then if } m > n \geq n_0 \text{ then } \left| \frac{1}{m^2} - \frac{1}{n^2} \right| < \frac{1}{n} < \frac{1}{\epsilon} = \epsilon.
\]

4. Use the definition to prove that the sequence \( \left\{ (-1)^n \right\} \) is not a Cauchy sequence.
Prove: \( \exists \epsilon > 0 \), \( \forall n_0 \in \mathbb{N} \), \( \exists m \geq n_0 \) and \( \exists n \geq n_0 \) such that \( \left| (-1)^m - (-1)^n \right| \geq \epsilon \).

Let \( n_0 \in \mathbb{N} \). Choose \( \epsilon = 2 \). Then \( n = n_0 \geq n_0 \) and \( m = n_0 + 1 \geq n_0 \), and \( \left| (-1)^m - (-1)^n \right| = \left| (-1)^{n_0} - (-1)^{n_0+1} \right| = 2 \geq \epsilon \).

5. Use the definition to prove that the sequence \( \left\{ n \right\} \) is not a Cauchy sequence.
Prove: \( \exists \epsilon > 0 \), \( \forall n_0 \in \mathbb{N} \), \( \exists m \geq n_0 \) and \( \exists n \geq n_0 \) such that \( |m - n| \geq \epsilon \).

Let \( n_0 \in \mathbb{N} \). Choose \( \epsilon = 1 \). Then \( n = n_0 \geq n_0 \) and \( m = n_0 + 1 \geq n_0 \), and \( |m - n| = |n_0 - (n_0 + 1)| = 1 \geq \epsilon \).

6. If \( \left\{ a_n \right\} \) has two subsequences that converge to different limits, then prove that \( \left\{ a_n \right\} \) diverges.

Theorem: If \( \left\{ a_n \right\} \) converges, then every subsequence of \( \left\{ a_n \right\} \) converges to \( \lim_{n \to \infty} a_n \).

An equivalent statement is:
If every subsequence of \( \left\{ a_n \right\} \) does not converge to \( \lim_{n \to \infty} a_n \), then \( \left\{ a_n \right\} \) diverges.

Now \( \left\{ a_n \right\} \) has two subsequences that converge to different limits \( \implies \) every subsequence of \( \left\{ a_n \right\} \) does converge to \( \lim_{n \to \infty} a_n \). Hence, by the Theorem above, the sequence \( \left\{ a_n \right\} \) diverges.

7. Let \( A \) be a nonempty subset of \( \mathbb{R} \) that is bounded below and let \( \beta = \inf A \). If \( \beta \notin A \), prove that \( \beta \) is a limit point of \( A \).

Let \( \epsilon > 0 \). We will show that \( N_{\epsilon}(\beta) \cap A \setminus \left\{ \beta \right\} \neq \emptyset \); that is, \( N_{\epsilon}(\beta) \cap A \neq \emptyset \) (since \( \beta \notin A \implies A \setminus \left\{ \beta \right\} = A \)). Then, since \( \beta < \beta + \epsilon, \beta + \epsilon \) is not a lower bound of \( A \) (since \( \beta = \inf A \)). \( \implies \exists a \in A \) such that \( \beta < a < \beta + \epsilon \). Thus \( \beta + \epsilon \) is an upper bound of \( A \). Hence, \( \beta \) is a limit point of \( A \).