Do the following.

**SECTION 7.1 EXERCISES** (page 495) Problems 2 and 3

**SECTION 7.2 EXERCISES** (page 509) Problems 4 and 5

1. Let \( T \in \mathcal{L}(V) \). A nonzero vector \( x \) in \( V \) is a generalized eigenvector of \( T \) corresponding to the eigenvalue \( \lambda \) if \((T - \lambda I)^p(x) = 0\) for some positive integer \( p \).
   Note: if \( p \) is the smallest positive integer such that \((T - \lambda I)^p(x) = 0\), then \((T - \lambda I)^{p-1}(x)\) is an eigenvector of \( T \).

2. Let \( T \in \mathcal{L}(V) \) and let \( x \) in \( V \) be a generalized eigenvector of \( T \) corresponding to the eigenvalue \( \lambda \).
   If \( p \) is the smallest positive integer such that \((T - \lambda I)^p(x) = 0\), then the ordered set \( \{ (T - \lambda I)^{p-1}(x), (T - \lambda I)^{p-2}(x), \ldots, (T - \lambda I)(x), x \} \)
   is called a cycle of generalized eigenvectors of \( T \) corresponding to the eigenvalue \( \lambda \). The length of the cycle, \( \{ (T - \lambda I)^{p-1}(x), (T - \lambda I)^{p-2}(x), \ldots, (T - \lambda I)(x), x \} \), is \( p \).
   Note:
   (1) the initial vector of a cycle of generalized eigenvectors of a linear operator \( T \) is the only eigenvector of \( T \) in the cycle.
   (2) If \( x \) is an eigenvector of \( T \) corresponding to the eigenvalue \( \lambda \) then the set \( \{ x \} \) is a cycle of generalized eigenvectors of \( T \), corresponding to the eigenvalue \( \lambda \), of length 1.

3. The generalized eigenspace of \( T \) corresponding to the eigenvalue \( \lambda \) of \( T \), denoted by \( K_\lambda \), is the set \( \{ x \in V | (T - \lambda I)^p \text{ for some positive integer } p \} \).
   Note \( E_\lambda \subseteq K_\lambda \).

4. If \( \lambda \) is an eigenvalue of \( T \) with multiplicity \( m \), then
   (a) \( \dim(K_\lambda) = m \).
   (b) \( K_\lambda = N((T - \lambda I)^m) \).

5. (DOT Diagram) Let \( r_j \) denote the number of dots in the \( j \)th row of the dot diagram of \( T_i \), the restriction of \( T \) to \( K_\lambda \). Then
   (a) \( r_1 = \dim(V) - \text{rank}(T - \lambda I) \)
   (b) \( r_j = \text{rank}((T - \lambda I)^{j-1}) - \text{rank}((T - \lambda I)^j) \) \( (j > 1) \).

6. INTERPRETATION of a DOT diagram:
   The block corresponding to the dot diagram \( \bullet \bullet \bullet \bullet \) consists of a cycle of length 3 (\# of dots in column 1), a cycle of length 2 (\# of dots in column 2), and a cycle of length 1 (\# of dots in column 1).

1. For each of the following matrices \( A \), find a Jordan canonical form \( J \) and an invertible matrix \( Q \) such that \( J = Q^{-1}AQ \).
   \( (a) \ A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix} \; |A - \lambda I| = \begin{vmatrix} 3 - \lambda & 1 & -2 \\ -1 & 0 - \lambda & 5 \\ -1 & -1 & 4 - \lambda \end{vmatrix} = -(\lambda - 3)(\lambda - 2)^2 = 0. \)
   So \( \lambda = 2 \) and its multiplicity is 2 and \( \lambda = 3 \).
   \( E_3; \)
   \( A - 3I = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \).
A basis for $E_3$ is $\{(-1, 2, 1)\}$.

$b_{2}$:

$$A - 2I = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ 2 & 2 & -2 - \lambda \end{bmatrix} = \begin{bmatrix} 3 - 0 & 2 & -3 \\ 0 & 1 - 0 & 0 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$  

A basis for $E_2$ is $\{(1, -3, -1)\}$.

Now $\dim(K_2) = 2 = \text{multiplicity of } 2 \text{ in } |A - 2I|$.

$$(A - 2I)^2 = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ -2 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Thus, a basis for $(A - 2I)^2$ is $\{(-1, 2, 0), (1, 0, 2)\}$.

A basis for the generalized eigenspace for $K_2$ is either a union of two 1-cycles or a 2-cycle. However, it cannot be two 1-cycles because each cycle contains one eigenvector but a basis for $E_2$ contains one eigenvector, since $\dim(E_2) = 1$. Therefore, a basis for the generalized eigenspace for $K_2$ is a 2-cycle. This is confirmed by the following dot diagram:

$$r_2 = 4 - \text{rank}(A - 2I) = 3 - 2 = 1$$

In order to form a basis (2-cycle) for $K_2$, we now look for a nonzero generalized eigenvector $x$ such that $(A - 2I)(x) \neq 0$ but $(A - 2I)^2(x) \neq 0$. From the basis $\{(-1, 2, 0), (1, 0, 2)\}$ of $(A - 2I)^2$ given above, we choose $x = (-1, 2, 0)$ (although both of these work) and note that $$\begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}.$$ So, the required 2-cycle is $\{(1, -3, -1), (-1, 2, 0)\}$.

The required Jordan canonical basis is $\beta' = \{(-1, 2, 1), (1, -3, -1), (-1, 2, 0)\}$. Hence, $Q = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Check: $Q^{-1} = \begin{bmatrix} -2 & -1 & 1 \\ -2 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ and $Q^{-1}AQ = J$.

(b) $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} -1 - \lambda & -1 & 0 \\ 0 & -1 - \lambda & -2 \\ 0 & 0 & -1 - \lambda \end{vmatrix} = -(1 + \lambda)^3 = 0.$$  

So $\lambda = -1$ and its multiplicity is 3.

$E_{-1}$:

$$A + I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. $$

A basis for $E_{-1}$ is $\{(1, 0, 0)\}$.

Now $\dim(K_{-1}) = 3 = \text{multiplicity of -1 in } A + I$.

$$(A + I)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Thus, a basis for $(A + I)^2$ is $\{1, x\}$. A basis for the generalized eigenspace for $K_{-1}$ is either a union of three 1-cycles, or a union of a 1-cycle and a 2-cycle, or a 3-cycle. However, it cannot be a union of three 1-cycles or a union of a 1-cycle and a 2-cycle because each of these cases requires two eigenvectors but a basis for $E_{-1}$ contains one eigenvector, since $\dim(E_{-1}) = 1$. Therefore, a basis for the generalized eigenspace for $K_{-1}$ is a 3-cycle. This is confirmed by the following dot diagram:
Note \((A + I)^3 = 1\) and \((A + I)^2 = 2\). In order to form a basis \((3\text{-cycle})\) for \(K_3\), we now look for a nonzero generalized eigenvector \(v\) such that \((A + I)^2(v) \neq 0\) and \((A + I)^3(v) = 0\).

\[(A + I)^2v = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix} \quad \text{and} \quad (A + I)v = \begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -2 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-2
\end{bmatrix}.

So, the required 3-cycle is \(\{2, -2x, x^2\}\). The required Jordan canonical basis is \(\beta' = \{(2, 0, 0), (0, -2, 0), (0, 0, 1)\}\). Hence, \(Q = \begin{bmatrix}
2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{bmatrix}\) and \(J = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{bmatrix}\).

Check: \(Q^{-1} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & -1
\end{bmatrix}\) and \(Q^{-1}AQ = J\).

\[(a) \quad A^2 = \begin{bmatrix}
2 & -4 & 2 \\
-2 & 0 & 1 \\
-2 & -6 & 3
\end{bmatrix}
\quad \text{and} \quad |A^2 - 2I| = \begin{vmatrix}
2 - \lambda & -4 & 2 \\
-2 & -\lambda & 1 \\
-2 & -6 & 3 - \lambda
\end{vmatrix} = (\lambda - 2)^2(\lambda - 4)^2 = 0.
\]

So \(\lambda = 2, -4\) and its multiplicity is 2, \(\lambda = 4\) and its multiplicity is 2.

For \(E_2\):

\[A - 2I = \begin{bmatrix}
0 & -4 & 2 \\
-2 & -4 & 1 \\
-2 & -6 & 3
\end{bmatrix} RREF \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -\frac{1}{2} \\
0 & 0 & 0
\end{bmatrix}.
\]

A basis for \(E_2\) is \(\{(0, 1, 2, 0), (2, 1, 0, 2)\}\).

For \(E_4\):

\[A - 4I = \begin{bmatrix}
-2 & -4 & 2 \\
-2 & -4 & 1 \\
-2 & -6 & 3
\end{bmatrix}
\quad \text{and} \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

A basis for \(E_4\) is \(\{(0, 1, 1, 1)\}\).

Now \(\dim(K_4) = 2\) with multiplicity 4 in \(|A - 4I|\).

\[(A - 4I)^2 = \begin{bmatrix}
4 & 8 & -4 & -4 \\
4 & 4 & 0 & -4 \\
4 & 0 & 4 & -4 \\
4 & 8 & -4 & -4
\end{bmatrix} RREF \begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Thus, a basis for \((A - 4I)^2\) is \(\{(-1, 1, 1, 0), (1, 0, 0, 1)\}\).

A basis for the generalized eigenspace for \(K_4\) is either a union of two 1-cycles or a 2-cycle. However, it cannot be two 1-cycles because each cycle contains one eigenvector but a basis for \(E_4\) contains one eigenvector, since \(\dim(E_2) = 1\). Therefore, a basis for the generalized eigenspace for \(K_4\) is a 2-cycle. This is confirmed by the following dot diagram:

\[r_1 = 4 - \text{rank}(A - 4I) = 4 - 3 = 1\] and \(r_2 = \text{rank}(A - 4I) - \text{rank}((A - 4I)^2) = 3 - 2 = 1\). In order to form a basis (2-cycle) for \(K_4\), we now look for a nonzero generalized eigenvector \(x\) such that \((A - 4I)(x) \neq 0\) but \((A - 2I)^2(x) \neq 0\). From the basis \(\{(-1, 1, 1, 0), (1, 0, 0, 1)\}\) of \((A - 4I)^2\) given above, we choose \(x = (-1, 1, 1, 0)\) (although both of these work) and note that \(A - 4I = \ldots \)
The required Jordan canonical basis is \( \beta' = \{(0, -1, -1, -1), (0, 1, 1, 0)\} \). Hence, \( Q = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \) and \( \lambda = 2 \) and its multiplicity is 2.

\( E_3: \) (d) \( A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix} \). \( |A - \lambda I| = (\lambda - 2)^2(\lambda - 3)^2 = 0. \)

So \( \lambda = 2 \) and its multiplicity is 2, \( \lambda = 3 \) and its multiplicity is 2.

\( A - 3I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \) with \( \text{RREF} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

A basis for \( E_3 \) is \( \{(1, 1, 1, 0), (0, 0, 0, 1)\} \).

\( E_2: \)

\( A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \) with \( \text{RREF} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

A basis for \( E_2 \) is \( \{(1, 0, 0, 0)\} \).

Now \( \dim(K_2) = 2 = \text{multiplicity of 2 in } |A - 2I|. \)

\( (A - 2I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \), \( \text{RREF} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

Thus, a basis for \( (A - 2I)^2 \) is \( \{(0, 0, 1, -1)\} \).

A basis for the generalized eigenspace for \( K_2 \) is either a union of two 1-cycles or a 2-cycle. However, it cannot be two 1-cycles because each cycle contains one eigenvector but a basis for \( E_2 \) contains one eigenvector, since \( \dim(E_2) = 1 \). Therefore, a basis for the generalized eigenspace for \( K_2 \) is a 2-cycle. This is confirmed by the following dot diagram:

\( r_1 = 4 - \text{rank}(A - 2I) = 4 - 3 = 1 \) and \( r_2 = \text{rank}(A - 2I) - \text{rank}((A - 2I)^2) = 3 - 2 = 1 \). In order to form a basis (2-cycle) for \( K_4 \), we now look for a nonzero generalized eigenvector \( x \) such that \( (A - 2I)(x) \neq 0 \) but \( (A - 2I)^2(x) \neq 0 \). From the basis \( \{(0, 1, 0, -1)\} \) of \( (A - 2I)^2 \) given above, we let \( x = (0, 1, 0, -1) \) and note that \( A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \) with \( \text{RREF} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

So, the required 2-cycle is \( \{(0, 1, 0, 0), (0, 1, 0, -1)\} \). The required Jordan canonical basis is \( \beta' = \{(1, 1, 1, 0), (0, 0, 0, 1), (1, 0, 0, 0), (0, 1, 0, -1)\} \). Hence, \( Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \) and

\( J = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \).