1. Use a triple integral in cylindrical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
   Ans. $2\int_0^{2\pi} \int_0^a (\int_0^{\sqrt{a^2-r^2}} dz) rdr \, d\theta = \frac{4}{3}\pi a^3$

2. Use cylindrical coordinates to solve the following problems.
   (a) Find the volume of the solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane. Ans. $\frac{\pi a^2}{2}$
   (b) A cylindrical hole of radius $a$ is bored through the center of a solid sphere of radius $2a$. Find the volume of the hole. Ans. $2\int_0^{2\pi} \int_0^{a} (\int_0^{\sqrt{4a^2-r^2}} dz) r dr \, d\theta = 4\sqrt{3}\pi a^3$
   (c) Find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$. Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\int_0^{\frac{1}{2}} (\int_0^{\sqrt{1-\frac{r^2}{4}}} dz) r dr] \, d\theta = \frac{\pi}{2}$
   (d) Find the volume of the region bounded above by the plane $z = x$ and below by the paraboloid $z = x^2 + y^2$. Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\int_0^{\sin \theta} (\int_0^{\sqrt{1-\frac{r^2}{2}}} dz) r dr] \, d\theta = \frac{\pi}{2}$
   (e) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$ ($a > 0$). Ans. $\int_0^{\pi} [\int_0^{a} (\int_0^{\sqrt{a^2-r^2}} dz) r dr] \, d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7)$
   (f) Find the volume of the region inside the cylinder $r = a \sin \theta$ which is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the ellipsoid $\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{2a^2} = 1$ where $b < a$. Ans. $\int_0^{\pi} [\int_0^{a \sin \theta} (\int_0^{\frac{a^2-r^2}{b^2}} dz) r dr] \, d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7) = \frac{1}{5}a^2(a-b)(3\pi-4)$

3. Evaluate $\iint_R dA$, where $R$ is the trapezoidal region with vertices given by $(0,0), (5,0), (\frac{5}{2}, \frac{5}{2}),$ and $(\frac{5}{2}, -\frac{5}{2})$, by making the change of variables $x = 2u + 3v$ and $y = 2u - 3v$. Note $u = \frac{x+y}{5}$ and $v = \frac{x-y}{3}$. The region in the uv-plane is a square with vertices $(0,0), (\frac{5}{2}, 0), (\frac{5}{2}, \frac{5}{2})$, and $(\frac{5}{2}, -\frac{5}{2})$. Ans. $\frac{5}{4} \times \frac{5}{2} = \frac{25}{8}$

4. Let $R$ be the region bounded by the line $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$. Evaluate $\int \int_R 3xydxdy$ by making the change of variables $x = \frac{1}{2}(2u + v)$, $y = \frac{1}{2}(u - v)$. We note that $J = \frac{1}{2}$ and $u = x + y$ and $v = x - 2y$. From the given equations, it follows that $1 \leq u = x + y \leq 4$ and $-4 \leq v = x - 2y \leq 0$. Thus $\int \int_R 3xydxdy = \int_1^4 \int_{-4}^0 3\left(\frac{1}{2}(2u + v)\right)\left(\frac{1}{2}(u - v)\right) dv \, du = 4$.

Alternatively, the given region is a rectangle with vertices $\left(-\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right)$ and the corresponding region in the uv-plane is the rectangle with vertices $(1,0), (4,0), (1,-4),$ and $(4,-4) = [1,4] \times [-4,0]$ Ans. $\int_1^4 \int_{-4}^0 3\left(\frac{1}{2}(2u + v)\right)\left(\frac{1}{2}(u - v)\right) \, dv \, du = 4$

5. Let $R$ be the region bounded by the square with vertices $(0,1), (1,2), (2,1),$ and $(1,0).$ Evaluate the integral $\int \int_R (x+y)^2 \sin^2(x-y)dxdy$ by making the change of variables $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. The corresponding region in the uv-plane is the square with vertices $(1,1), (3,1), (1,-1),$ and $(3,-1)$. $J = \frac{1}{2}$

$\int \int_R (x+y)^2 \sin^2(x-y)dxdy = \int_{-1}^1 \int_{-1}^1 (u^2 \sin^2 v\frac{1}{2} du)dv = \frac{13}{6}(2 - \sin 2)$


6. Let \( R \) be the region bounded by the parallelogram with vertices \((0, 0), (4, 0), (3, 3), \) and \((7, 3)\).
Evaluate the integral \( \int_{R} y(x - y) dx dy \) by making the change of variables \( x = u + v, y = u \).
Ans. 36

7. Let \( B \) be the square with vertices at \((0, 1), (1, 0), (2, 1), \) and \((1, 2)\).
Evaluate \( \int_{B} 60xy \ dx \ dy \)
by making the change of variables \( x = \frac{1}{2}(u + v), y = -\frac{1}{2}(u - v) \).
Ans. 120

8. Use a triple integral in spherical coordinates to find the volume of the sphere \( x^2 + y^2 + z^2 = a^2 \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = \frac{4}{3} \pi a^3 \).

9. Use a triple integral in spherical coordinates to find the volume of the sphere \( x^2 + y^2 + z^2 = 2z \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = \frac{4}{3} \pi \).

10. Find the volume of the region bounded by the sphere \( \rho = a \) and the cone \( \phi = \alpha \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{\alpha}^{\pi} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = \frac{4}{3} \pi a^3 (1 - \cos \alpha) \).

11. If \( 0 < b < a \) and \( 0 < \alpha < \pi \), find the volume of the region bounded by the concentric spheres \( \rho = b, \rho = a \) and the cone \( \phi = \alpha \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{\alpha}^{\pi} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = \frac{2\pi (\alpha - b^3) (1 - \cos \alpha)}{4} \).

12. Find the volume of the solid region bounded below by the upper nappe of the cone \( z^2 = x^2 + y^2 \)
and above by the sphere \( x^2 + y^2 + z^2 = 9 \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{3} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = 9\pi (2 - \sqrt{2}) \).

13. Evaluate \( \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \int_{E} 16z \ dx \ dy \), where \( E \) is the upper half of the sphere \( x^2 + y^2 + z^2 = 1 \).
Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} (16 \cos \phi) \rho^2 \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta = 4\pi \).

14. Convert \( \int_{0}^{a} \int_{0}^{\sqrt{a^2 - y^2}} \int_{y}^{\sqrt{a^2 - y^2}} (x^2 + y^2 + z^2) \ dx \ dz \ dy \) into an integral in
(a) spherical coordinates. Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta \).

(b) cylindrical coordinates. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^2 \sin \phi \ \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta \).

15. Convert \( \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_{0}^{\sqrt{a^2 - x^2 - y^2}} x \ dz \ dy \ dz \)
into an integral in
(a) spherical coordinates. Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} (\rho^3 \cos \phi + \rho^3 \phi \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta \).

(b) cylindrical coordinates. Ans. \( \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} (\rho^3 + \rho^3 \phi \rho \ \rho \ \phi \ \phi d\rho \ \phi d\phi d\theta \).

16. Evaluate \( \int_{C} (x - 3y) \ dx \), where \( C \) is the line segment from \((0, 0)\) to \((1, 2)\) and \( C \) is parameterized as:
(a) \( x = t, \ y = 2t, 0 \leq t \leq 1 \).
(b) \( x = \sin t, \ y = 2 \sin t, 0 \leq t \leq \frac{\pi}{2} \).

17. Evaluate \( \int_{C} (x^2 - y + 3z) \ dx \), where \( C \) is the line segment from \((0, 0, 0)\) to \((1, 2, 1)\). [ Answer. \( \int_{0}^{1} (t^2 - 2t + 3t) \sqrt{6} \ dt = \frac{5\sqrt{6}}{6} \)].

18. Evaluate \( \int_{C} x \ dx \), where \( C \) is the line segment from \((0, 0)\) to \((1, 1)\) and \( y = x^2 \) from \((1, 1)\) to \((0, 0)\). [ Answer. \( \int_{0}^{1} \sqrt{2} \ dt = \int_{0}^{1} (1 - t) \sqrt{1 + 4(1 - t)^2} \ dt = \frac{\sqrt{2}}{2} + \frac{1}{2} (5^2 - 1) \)].

19. Evaluate the line integrals \( \int_{C} (x - 3y) \ dx \), and \( \int_{C} (x - 3y) \ dy \), if \( C \) is the part of the parabola \( x = y^2 \)
that joins the points \((1, 1)\) and \((4, 2)\) [ Answer. \( \int_{1}^{2} (t^2 - 3t) 2t \ dt = \frac{-13}{2} \) and \( \int_{1}^{2} (t^2 - 3t) 2t \ dt = \frac{-13}{6} \)].
20. Evaluate the line integral $\int_C (y^2 dx - x^2 dy)$ along the two curves given below:

(a) $C_1$: The parabola $x = t$, $y = t^2$ joining the two points $(0, 0)$ and $(2, 4)$. [Answer. $\int_0^2 (t^4 - 2t^3) dt = \frac{28}{3}$].

(b) $C_2$: The line $x = t$, $y = 2t$ joining the two points $(0, 0)$ and $(2, 4)$. [Answer. $\int_0^2 2t^3 dt = \frac{16}{3}$].

21. Evaluate the line integral $\int_C xy^2 dx - (x + y) dy$, where $C$ is

(a) The straight line segment from $(0, 0)$ and $(1, 2)$. [Answer. $\int_0^1 (4t^3 - 6t) dt = -2$].

(b) The parabolic path from $(0, 0)$ and $(2, 4)$. [Answer. $\int_0^1 (4t^5 - 4t^2 - 8t^3) dt = -\frac{8}{3}$].

(c) The broken line from $(0, 0)$ to $(1, 0)$ to $(1, 2)$. [Answer. $-\int_0^1 (1 + t) dt = -4$].

22. Evaluate the line integral $\int_C xy^2 dx - (x + y) dy$, where $C$ is the broken line joining the points $(0, 0)$, $(1, 1)$, $(2, 1)$ in this order.

23. Evaluate the line integral $\int_C \frac{dx}{y^2} + \frac{dy}{x^2}$, where $C$ is the part of the hyperbola $xy = 4$ from $(1, 4)$ to $(4, 1)$. [Answer. $\int_1^4 \left(\frac{1}{4} - \frac{1}{x^2}\right) dt = 0$].

24. Evaluate the line integral $\int_C xdx + x^2 dy$ from $(−1, 0)$ to $(1, 0)$.

(a) Along the $x$-axis. [Answer. $\int_{−1}^1 t dt = 0$].

(b) Along the semicircle $y = \sqrt{1 - x^2}$ [Answer. $\int_0^1 (−\sin t + 1 - \sin^2 t) dt = 0$].

(c) Along the broken line from $(-1, 0)$ to $(0, 1)$ to $(1, 1)$. [Answer. $\int_{−1}^1 (t + t^2) dt + \int_0^1 t dt - \int_{−1}^1 dt = -\frac{1}{6} + \frac{1}{2} - 1 = -\frac{2}{3}$].

25. Evaluate the line integral $\int_C ydx + (x + 2y) dy$ from $(1, 0)$ to $(0, 1)$, where $C$ is

(a) the arc of the circle $x = \cos t$, $y = \sin t$; [Answer. $\int_0^\frac{\pi}{2} (\cos^2 t - \sin^2 t + 2 \sin t \cos t) dt = 1$].

(b) the straight line segment $y = 1 - x$; [Answer. $\int_1^0 (-1) dx = 1$].

(c) the broken line from $(1, 0)$ to $(1, 1)$ to $(0, 1)$. [Answer. $\int_1^0 (1 + 2y) dy + \int_0^1 dx = 1$].

26. Evaluate $\int_C (3x + 4y) dx + (2x + 3y^2) dy$, where $C$ is the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$. [Answer. $x = 2 \cos t$, $y = 2 \sin t$, $0 \leq t \leq 2\pi$, $\int_0^{2\pi} ((24 \sin^2 t - 12 \sin t \cos t + 8 - 24 \sin^2 t) dt = -8\pi]$.

27. Evaluate $\int_C (x + 2) ds$, where $C$ is the curve represented by $c(t) = ti + \frac{1}{3} t^2 j + \frac{1}{2} t^2 k$, $0 \leq t \leq 6\pi$.

28. Evaluate $\int_C \mathbf{F} \cdot ds$, if $\mathbf{F} = xy^2 \mathbf{i} + x^2 \mathbf{j} - (y - x) \mathbf{k}$, $C$ is the curve $c(t) = ti + t^2 \mathbf{j} + t^3 \mathbf{k}$, $0 \leq t \leq 1$.

29. Evaluate $\int_C \mathbf{F} \cdot ds$, if $\mathbf{F} = y^2 \mathbf{i} + x^2 \mathbf{j} - (y - x) \mathbf{k}$, $C$ is the straight line segment from the point $(1,0,-1)$ to the point $(3,3,5)$.

30. Evaluate $\int_C \mathbf{F} \cdot ds$, if $\mathbf{F}(x,y) = (x + y) \mathbf{i} + (y^2 - x) \mathbf{j}$, where $C$ is the closed curve that begins at $(1,0)$, proceeds along the upper half of the unit circle to $(-1,0)$, and returns to $(1,0)$ along the $x$-axis.

31. (a) Evaluate $\int_S xy^4 ds$, where $C$ is the right half of the circle, $x^2 + y^2 = 1$. [Answer. $\int_0^{\frac{\pi}{2}} (4 \cos t)(4 \sin t)^4 (4 dt = \frac{8192}{3})$].

(b) Evaluate $\int_S 4x^3 ds$, were $C$ is the line segment from $(-2,-1)$ to $(1,2)$. [Answer. $\int_{−2}^1 4(-2 + 3t)^3 \sqrt{1^2 dt = -15\sqrt{2}}$].

(c) Evaluate $\int_S 4x^3 ds$, were $C$ is the line segment from $(1,2)$ to $(−2,−1)$. [Answer. $\int_0^1 4(1 - 3t)^3 \sqrt{1^2 dt = -15\sqrt{2}}$].
32. Evaluate $\int_C xds$ for each of the following curves. (a) $C_1$: $y = x^2, -1 \leq x \leq 1$ (b) $C_2$: The line segment from (-1,1) to (1,1). (c) $C_3$: The line segment from (1,1) to (-1,1). [Answer: 0 for each part (a)-(c)].

33. Evaluate $\int_C xyds$, where $C$ is the helix given by, $x = \cos t, y = \sin t, z = 3t, 0 \leq t \leq 4\pi$.

34. Evaluate $\int_C \sin(\pi y)dy + yx^2dx$, where $C$ is the line segment from (0,2) to (1,4). [Answer: $\frac{7}{6}$].

35. Evaluate $\int_C \sin(\pi y)dy + yx^2dx$, where $C$ is the line segment from (1,4) to (0,2). [Answer: $-\frac{7}{6}$].

36. Evaluate $\int_C ydx + xdy + zdx$, where $C$ is given by, $x = \cos t, y = \sin t, z = t^2, 0 \leq t \leq 2\pi$. [Answer: $\int_0^{2\pi} (-\sin^2 t + \cos^2 t + 2t^3)dt = 8\pi^4$].

37. Evaluate $\int_C (2xy + x^2)dy$ if:
   (a) $C$ consists of the line segments from (3,1) to (5,1) and from (5,1) to (5,6).
   (b) $C$ is the line segments from (3,1) to (5,6).
   (c) $C$ is the part of the parabola $x = 2t + 1, y = 2t^2 - t, 1 \leq t \leq 2$. [Answer: 141, for (a), (b), (c)]

38. Find a potential function for
   (a) $\mathbf{F}(x,y) = (2xy + 24x)i + (x^2 + 16y)j$. (Answer: $x^2y + 12x^3 + 16y + C$)
   (b) $\mathbf{F}(x,y) = 2xyi + (x^2 - y)j$. (Answer: $x^2y - \frac{x^2}{2} + C$)
   (c) $\mathbf{F}(x,y,z) = 2xyi + (x^2 + z^2)j + 2yzk$. (Answer: $x^2y + z^2y + C$)
   (d) $\mathbf{F}(x,y,z) = (y \cos x + 2x^2y)\mathbf{i} + (\sin x + x^2e^y + 4)\mathbf{j}$. (Answer: $y \sin x + x^2e^y + 4 + C$)
   (e) $\mathbf{F}(x,y) = (x^2 - xy)\mathbf{i} + (y^2 + xy)\mathbf{j}$.
   (f) $\mathbf{F}(x,y) = (2xe^{xy} + x^2y^3)\mathbf{i} + (3xe^{xy} + 2y)\mathbf{j}$. (Answer: $x^2e^{xy} + y^2 + C$)
   (g) $\mathbf{F}(x,y,z) = 2xy^3z^4\mathbf{i} + (3x^2y^2z^4)\mathbf{j} + 4x^2y^3z^3\mathbf{k}$. (Answer: $x^2y^3z^4 - C$)
   (h) $\mathbf{F}(x,y,z) = 2x^2y - 2xz^2\mathbf{i} + (3 + 2ye^z - x^2\sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$. (Answer: $y^2e^z - 2xz^2 + x^2\cos y + 3y + C$)

39. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is a piecewise smooth curve from (-1,4) to (1,2) and $\mathbf{F}(x,y) = 2xyi + (x^2 - y)j$. (Answer: $x^2y - \frac{x^2}{2} + C$, 4)

40. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is a piecewise smooth curve from (1,1,0) to (0,2,3) and $\mathbf{F}(x,y,z) = 2xyi + (x^2 + z^2)j + 2yzk$. (Answer: $x^2y + z^2y + C$, 17)

41. Use Green’s theorem to evaluate the line integral $\oint_C [xydx + x^2y^3dy]$, where $C$ is the triangle with vertices (0,0), (1,0), and (1,2).
   (Answer: $P = xy, Q = x^2y^3$, the region enclosed is $0 \leq x \leq 1, 0 \leq y \leq 2x$, $\int_0^1 \int_0^{2x} (Q_x - P_y)dydx = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \frac{2}{3}$).

42. Use Green’s theorem to evaluate the line integral $\oint_C [y^3dx + x^3dy]$, where $C$ is the circle of radius 2 centered at (0,0).
   (Answer: $P = y^3, Q = x^3$, the region enclosed is $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$, $\int_0^{2\pi} \int_0^2 (Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2 (3r^2 \cos^2 \theta - 3r^2 \sin^2 \theta)rdrd\theta = 0$).

43. Use Green’s theorem to evaluate the line integral $\oint_C [(-2xy+y^2)dx + x^2dy]$, where $C$ is the boundary of the region $R$ enclosed by $y = 4x$ and $y = 2x^2$.
   (Answer: $P = -2xy + y^2, Q = x^2$, the region enclosed is $0 \leq x \leq 2, 2x^2 \leq y \leq 4x$, $\int_0^2 \int_{2x^2}^{4x} (Q_x - P_y)dydx = \int_0^2 \int_{2x^2}^{4x} (4x - 2y)dydx = \frac{32}{9}$).
44. Evaluate the line integral \( \int_C (3x - y)dx + (x + 5y)dy \) around the unit circle \( x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \).
   (Answer: \( P = 3x - y, Q = x + 5y \), the region enclosed is \( 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2 \), \( \int_0^{2\pi} \int_0^1 (Qx - P_y)rdrd\theta = \int_0^{2\pi} \int_0^1 (1 - (-1))rdrd\theta = 2\pi \).

45. Evaluate the line integral \( \int_C (e^{-x^2} + y^2)dx + (\ln y - x^2)dy \), where \( C \) is the square with vertices \((0, 0), (1, 0), (1, 1), (0, 1)\).
   (Answer: \( P = e^{-x^2} + y^2, Q = \ln y - x^2 \), the region enclosed is \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), \( \int_0^1 \int_0^1 (Qx - P_y)dydx = \int_0^1 \int_0^1 (1 - 2x - 2y)dydx = -2 \).

46. Evaluate the line integral \( \int_C (2y + \sqrt{1 + x^2})dx + (5x - e^y)dy \) around the circle \( x^2 + y^2 = 4 \).
   (Answer: \( P = 2y + \sqrt{1 + x^2}, Q = 5x - e^y \), the region enclosed is \( 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2 \), \( \int_0^{2\pi} \int_0^2 (Qx - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2 (5 - 2)rdrd\theta = 12\pi \).

47. If \( R \) is any region to which Green’s Theorem is applicable, show that the area of \( R \) is given by the formula:

\[ A = \frac{1}{2} \int_C -ydx + xdy, \quad \text{or} \quad A = \oint_C xdy, \quad \text{or} \quad A = \oint_C (-y)dx. \]

48. Use Green’s theorem to find the area of the region enclosed by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
   (Answer. Parametric equations for the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \( x = a \cos t, y = b \sin t, 0 \leq t \leq 2\pi \).
   Area = \( \frac{1}{2} \int_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} (-b \sin t)(-a \sin t dt) + (a \cos t)(b \cos t dt) = \pi ab \).

49. Use Green’s theorem to find the area of the region enclosed by the circle \( x^2 + y^2 = a^2 \).
   (Answer. Parametric equations for the circle \( x^2 + y^2 = a^2 \) is \( x = a \cos t, y = a \sin t, 0 \leq t \leq 2\pi \).
   Area = \( \frac{1}{2} \int_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} (-a \sin t)(-a \sin t dt) + (a \cos t)(a \cos t dt) = \pi a^2 \).

50. Use Green’s theorem to find the area of the region enclosed by \( y = x^2 \) and \( y = x + 2 \).
   (Answer. Parametric equations for the boundary of the region: \( C_1 : x = t, y = t^2, -1 \leq t \leq 2 \); \( C_2 : x = -3t + 2, y = -3t + 4 \) Area = \( \frac{1}{2} \int_{C_1} -ydx + xdy = \frac{1}{2} \int_{C_1} ((-t^2)(dt) + (t)(2dt)) + \frac{1}{2} \int_{C_2} ((3t - 4)(-3dt) + (-3t + 2)(-3dt)) = \pi \).

51. Use Green’s theorem to find the area of the region enclosed by \( y = x^2 - 1 \) and \( y = 0 \).
   (Answer. Parametric equations for the boundary of the region: \( C_1 : x = t, y = 1 - t^2, -1 \leq t \leq 1 \); \( C_2 : x = -t, y = 0, -1 \leq t \leq 1 \) Area = \( \frac{1}{2} \int_{C_1} -ydx + xdy + \frac{1}{2} \int_{C_2} -ydx + xdy = \frac{1}{2} \int_{-1}^1 [((-1 - (-t^2))(dt) + (t)(-2dt)) + \frac{1}{2} \int_{-1}^1 (0)(dt) + (-t)(0dt)]. \)

52. Use Green’s theorem to find the area under one arc of the cycloid \( x = 2\pi t - \sin 2\pi t \) and \( y = 1 - \cos 2\pi t; 0 \leq t \leq 1 \).
   (Answer. Parametric equations for the boundary of the region: \( C_1 : x = t, y = 0, 0 \leq t \leq 2\pi \); \( C_2 : x = 2\pi t - \sin 2\pi t, y = 1 - \cos 2\pi t, 0 \leq t \leq 1 \) Area = \( \frac{1}{2} \int_{C_1} -ydx + xdy = \frac{1}{2} \int_{C_2} -ydx + xdy = \frac{1}{2} \int_{-1}^1 ((0)(dt) + (t)(0dt)) + \frac{1}{2} \int_{-1}^1 ((-2(2 \pi t - \sin 2\pi t)(2\pi - 2\pi \cos 2\pi t)dt) + (2\pi t - \sin 2\pi t)(2\pi \sin 2\pi t dt)) = 3\pi. \)

53. Determine the surface given by the parametric equations \( r(u, v) = ui + u \cos vj + u \sin vk \).

54. Give parametric representations for each of the following surfaces.

(a) The elliptic paraboloid \( x = 5y^2 + 2z^2 - 10 \).
   \( x = 5y^2 + 2z^2 - 10, y = y, z = z \)

(b) The elliptic paraboloid \( x = 5y^2 + 2z^2 - 10 \) that is in front of the \( yz \)-plane.
   \( x = 5y^2 + 2z^2 - 10, y = y, z = z, x \geq 0 \).
(c) The sphere \(x^2 + y^2 + 2z^2 = 30.\)
\[x = \sqrt{30} \sin \phi \cos \theta, \quad y = \sqrt{30} \sin \phi \sin \theta, \quad z = \sqrt{30} \cos \phi; \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.\]

(d) The cylinder \(y^2 + 2z^2 = 25.\)
\[x = x, \quad y = 5 \sin \theta, \quad z = 5 \cos \theta, \quad 0 \leq \theta \leq 2\pi.\]