EXERCISES  Page 390  1-5, 8, 9,  13, 14,  17, 19, 21, 22, 23-25

1. Use a triple integral in cylindrical coordinates to find the volume of the sphere \( x^2 + y^2 + z^2 = a^2 \).  
   Ans. \( 2 \int_0^{2\pi} \int_0^a [r \int_0^{\sqrt{a^2 - r^2}} dz] r \, dr \, d\theta = \frac{4}{3} \pi a^3 \)

2. Use cylindrical coordinates to solve the following problems.
   (a) Find the volume of the solid bounded above by the paraboloid \( z = 1 - x^2 - y^2 \) and below by the xy-plane.  
      Ans. \( \int_0^{2\pi} \int_0^1 [r \int_0^{\sqrt{1-r^2}} dz] r \, dr \, d\theta = \frac{\pi}{2} \)
   (b) A cylindrical hole of radius \( a \) is bored through the center of a solid sphere of radius \( 2a \). Find the volume of the hole.  
      Ans. \( 2 \int_0^{2\pi} \int_0^a [r \int_0^{\sqrt{2a^2 - r^2}} dz] r \, dr \, d\theta = 4\sqrt{3} \pi a^3 \)
   (c) Find the volume of the region bounded above by the plane \( z = 2x \) and below by the paraboloid \( z = x^2 + y^2 \).  
      Ans. \( \int_0^{2\pi} \int_0^a [r \int_0^{\sqrt{a^2 - r^2}} dz] r \, dr \, d\theta = \frac{\pi}{2} \)
   (d) Find the volume of the region bounded above by the plane \( z = x \) and below by the paraboloid \( z = x^2 + y^2 \).  
      Ans. \( \int_0^{2\pi} \int_0^a [r \int_0^{\cos \theta} dz] r \, dr \, d\theta = \frac{\pi}{2} \)
   (e) Find the volume of the region bounded above by the sphere \( x^2 + y^2 + z^2 = 2a^2 \) and below by the paraboloid \( az = x^2 + y^2 \).  
      Ans. \( \int_0^{2\pi} \int_0^a [r \int_0^{\cos \theta} dz] r \, dr \, d\theta = \frac{\pi}{6} \pi a^3 (8\sqrt{2} - 7) \)
   (f) Find the volume of the region inside the cylinder \( r = a \sin \theta \) which is bounded above by the sphere \( x^2 + y^2 + z^2 = a^2 \) and below by the upper half of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \) where \( b < a \).  
      Ans. \( \int_0^{2\pi} \int_0^a [r \int_0^{\frac{\sqrt{a^2 - r^2}}{\sqrt{b^2 + r^2}} dz] r \, dr \, d\theta = \frac{1}{6} \pi a^3 (8\sqrt{2} - 7) = \frac{1}{6} a^2 (a - b)(3\pi - 4) \)

3. Evaluate \( \int_R \, dA \), where \( R \) is the trapezoidal region with vertices given by \( (0,0), (5,0), (\frac{5}{2}, \frac{5}{2}) \), and \( (\frac{5}{2}, -\frac{5}{2}) \), by making the change of variables \( x = 2u + 3v \) and \( y = 2u - 3v \).  
   Note \( u = \frac{x + y}{4} \) and \( v = \frac{x - y}{6} \). The region in the uv-plane is a square with vertices \( (0,0), (\frac{5}{4}, 0), (\frac{5}{4}, \frac{5}{4}) \), and \( (0, \frac{5}{4}) \).  
   Ans. \( \frac{5}{2} \times \frac{5}{2} = \frac{25}{4} \).

4. Let \( R \) be the region bounded by the line \( x - 2y = 0, x - 2y = -4, x + y = 4, \) and \( x + y = 1 \).  
   Evaluate \( \int_R 3x y \, dx \, dy \) by making the change of variables \( x = \frac{1}{2}(2u + v), y = \frac{1}{2}(u - v) \).  
   We note that \( J = \frac{1}{3} \) and \( u = x + y \) and \( v = x - 2y \). From the given equations, it follows that \( 1 \leq u = x + y \leq 4 \) and \( -4 \leq v = x - 2y \leq 0 \). Thus \( \int_R 3x y \, dx \, dy = \int_1^4 \int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v)) \, dv \, du = 4 \).
   Alternatively, the given region is a rectangle with vertices \( (-\frac{2}{3}, \frac{5}{3}), (0, \frac{5}{3}), (\frac{4}{3}, \frac{4}{3}), \) and \( (\frac{4}{3}, -\frac{4}{3}) \) and the corresponding region in the uv-plane is the rectangle with vertices \( (1,0), (4,0), (1,-4), \) and \( (4,-4) = [1,4] \times [-4,0] \).  
   Ans. \( \int_1^4 \int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v)) \, dv \, du = 4 \).

5. Let \( R \) be the region bounded by the square with vertices \( (0,1), (1,2), (2,1), \) and \( (1,0) \). Evaluate the integral \( \int_R (x + y)^3 \sin^2(x - y) \, dx \, dy \) by making the change of variables \( x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \).  
   The corresponding region in the uv-plane is the square with vertices \( (1,1), (3,1), (1,-1), \) and \( (3,-1) \).  
   \( J = \frac{1}{2} \)
   \( \int_R (x + y)^3 \sin^2(x - y) \, dx \, dy = \int_{-1}^1 \int_{-1}^1 \frac{1}{2} u^3 \sin^2 v u^2 \, du \, dv = \frac{13}{6}(2 - \sin 2) \).

6. Let \( R \) be the region bounded by the parallelogram with vertices \( (0,0), (4,0), (3,3), \) and \( (7,3) \). Evaluate the integral \( \int_R (y - x) \, dx \, dy \) by making the change of variables \( x = u + v, y = u \).  
   Ans. 36
7. Let $B$ be the square with vertices at $(0, 1)$, $(1, 0)$, $(2, 1)$, and $(1, 2)$.
Evaluate $\int \int_B 60 \, x \, y \, dx \, dy$
by making the change of variables $x = \frac{1}{2}(u + v)$, $y = -\frac{1}{2}(u - v)$.
Ans. 120

8. Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy$ into an integral in cylindrical coordinates.
$\int_0^{\pi/2} \left[ \int_0^3 \left( \int_r^{\sqrt{18-r^2}} (r^2 + z^2) \, dz \right) r \, dr \right] \, d\theta$

9. Convert $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{a+\sqrt{a^2-x^2-y^2}}^{a+\sqrt{a^2-x^2-y^2}} x \, dz \, dy \, dx$
into an integral in cylindrical coordinates. Ans. $\int_0^{2\pi} \int_0^a (\int_a^{a+\sqrt{a^2-r^2}} (r \cos \theta) \, dz) \, dr \, d\theta$. 