An Introduction to the Riemann Curvature Tensor and Differential Geometry

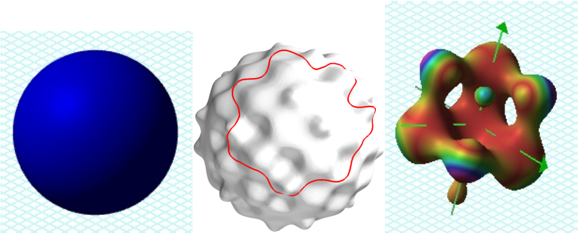
Corey Dunn

2010 CSUSB REU Lecture # 1

June 28, 2010

What is differential geometry?

In what way do the following objects differ?



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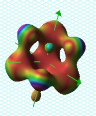
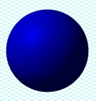
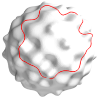
1.  has holes in it, whereas  and  don't.

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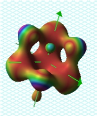
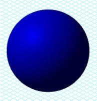
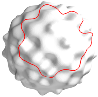
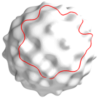
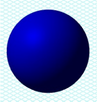
1.  has holes in it, whereas  and  don't.

2.  has ripples, whereas  seems to curve more uniformly.

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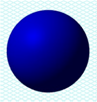
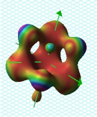
Topology is more concerned between differentiating between

 and .

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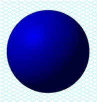

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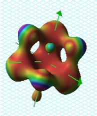

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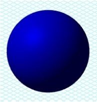

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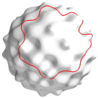

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Big Question:

What "quantities" are useful? And more importantly, what quantities are intrinsic to the surface, as opposed to objects which depend on our perspective?

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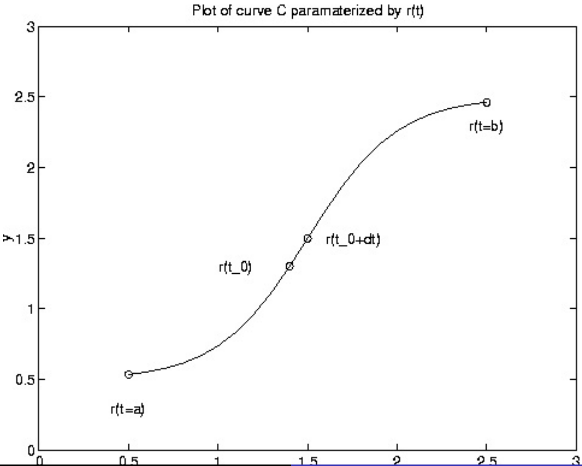
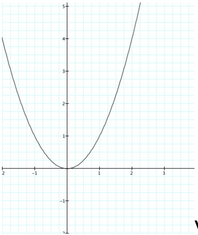
Arc length is intrinsic: no matter how we twist or rotate a curve, its arc length will always be the same.



A quantity which is *not* intrinsic. . .

Given a curve in $\mathbb{R}^2$, the “area under the curve” depends on how you arrange your axes:



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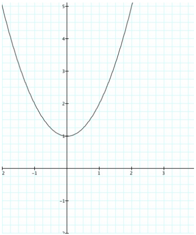

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What next?

- A short history of the development of curvature.
- A question about curvature to which I'd like an answer!

The beginnings of intrinsic curvature:

Johann Karl Friedrich Gauss (1777-1855)

Georg Friedrich Bernhard Riemann (1826-1866)

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Theorem (Gauss)

An inhabitant of a 2-dimensional surface can measure the curvature of their universe by using only a ruler and a protractor. This curvature is intrinsic to the surface.

That is: A 2-dimensional person may measure the curvature of the surface they reside in, *without leaving the surface*!

This is called the **Gaussian curvature** of a surface.

Where curvature came from, 2:

Riemann:

- To earn a university position, he had to
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- He developed what is known now as the *Riemann curvature tensor*, a generalization to the Gaussian curvature to higher dimensions.

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A reasonably non-technical definition:

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Key point: Curvature is measured at each point of a manifold, and is an object with distinct symmetries. Understanding how this object acts at any single point gives us insight to how *any* object could curve. (More on this in a moment.)

Summary:

To summarize:

- Gauss developed an intrinsic notion of curvature for 2-dimensional manifolds.
- Riemann extended this notion to higher dimensional manifolds.
- Mathematically technical objects such as manifolds, metrics, and curvature put observable geometric notions on firm mathematical footing.

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Recall the ingredients to computing curvature:

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In "English": Since the tangent space of a manifold at any point is just a vector space, the curvature tensor at that point is just a certain real valued function whose domain is a vector space.

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A reasonably non-technical definition:

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5. Can we generalize our efforts from the previous two questions to determine a *useful* basis for this vector space?

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- Perhaps a discussion of "geometric realizability": the process of finding a manifold with predescribed curvature properties. Also perhaps a discussion of "curvature homogeneity": a notion in differential geometry of recent interest.

Thank you!!!!!!

More to come tomorrow . . .