Worksheet on sequences and series

May 20, 2010

1 Sequences

1. What should a sequence be? Which of the following fall into what you think a sequence should be?
   
   (a) 1, 2, 3, 4, . . .
   (b) 1, 0, 2, 0, 4, 0, 8, 0, . . .
   (c) The set of real numbers greater than or equal to 0.
   (d) The set of all rational numbers (numbers which are quotients of integers).

2. Formulate a mathematical definition of what a sequence is. Then, use your definition to formulate the definition of a subsequence.

3. Formulate a definition of what it means for a sequence to converge.

4. Imagine a situation where you have a sequence of numbers converging to a real number, say \( \{ \frac{1}{n} \} \) converging to \( a = 0 \). Now imagine a function that has a domain large enough to eat all of the members of your sequence (as inputs), and what your sequence converges to, say \( f(x) = 3x + 2 \). In this example, we have

   \[
   \lim_{x \to 0} (3x + 2) = 2, \quad \text{and} \quad \lim_{n \to \infty} f \left( \frac{1}{n} \right) = 2.
   \]

   Would the limit on the right be any different if you considered a different sequence which converged to 0? Why or why not?

5. Prove: \( \lim_{x \to a} f(x) = L \) if and only if for every sequence \( a_n \) with \( a_n \) converging to \( a \) and \( a_n \neq a \) for all \( n \), we have \( \lim_{n \to \infty} f(a_n) = L \). (Hint: before you get started, ask yourself how this question is an extension of the last one.)

6. Smile quietly as to how limits of sequences can be phrased as limits of real numbers, so that all of our basic limit properties can now be invoked. For example, we now may use the squeeze theorem, basic limit properties, and everything else we’ve done using limits in the context of sequences.
7. Prove the following about bounded monotone sequences: Let \( a_n \) be a sequence which is bounded (i.e., there exists a real number \( M \) so that \( |a_n| < M \) for all \( n \)) and monotone (i.e., it is either increasing the whole time, or decreasing the whole time). Prove that \( a_n \) converges.

2 Infinite Series

8. Does it even make sense to add together infinitely many numbers? (Yes.) Formulate a definition of \( \sum_{n=1}^{\infty} a_n \) in terms of a limit of a certain sequence.

9. Evaluate \( \sum_{n=0}^{\infty} \frac{1}{2^n} \) using only your definition.

10. Prove the following basic facts about series. Assume \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge, and that \( k \) is a real number. Prove the following:

\[
\sum_{n=1}^{\infty} ka_n = k \sum_{n=1}^{\infty} a_n, \quad \text{and} \quad \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.
\]

11. True or False: If the series \( \sum_{n=1}^{\infty} a_n \) converges, then the sequence \( a_n \) converges.

12. True or False: If the sequence \( a_n \) converges, then the series \( \sum_{n=1}^{\infty} a_n \) converges.