Hi everyone! Here is your first homework assignment. I’ll be adding questions to it over the next week or so. I’ll announce in class (and change the website announcement) when it’s final. Good luck! These 10 questions are all that will be part of homework # 1! It’s due date is February 9th.

1. Recall that the product topology is defined on the topologies \( X \) and \( Y \) as follows: A subset \( U \subseteq X \times Y \) is open in \( X \times Y \) if for all \((x, y) \in U\), there exists \( U_X \) (open in \( X \)) and \( U_Y \) (open in \( Y \)) so that \((x, y) \in U_X \times U_Y \subseteq U\). Please show that the standard topology on \( \mathbb{R}^2 \) is the same topology as the product topology on \( \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \). That is, show that two topologies on the same space contain exactly the same open sets.

2. Recall the \( \varepsilon - \delta \) definition of continuity of a function \( f : X \rightarrow Y \), where \( X \subseteq \mathbb{R}^n \), and \( Y \subseteq \mathbb{R}^m \) (suppose that \( X \) and \( Y \) are both open subsets in the standard topology of each Euclidean space). We may endow \( X \) and \( Y \) with the subspace topology, and speak of the continuity of \( f \) as a function between topological spaces. Prove that the \( \varepsilon - \delta \) notion of continuity is the same as the topological definition of continuity.

3. Suppose \( f : X \rightarrow Y \), a function between topological spaces. Prove that \( f \) is continuous if and only if, for every closed set \( F \subseteq Y \), \( f^{-1}(F) \) is closed in \( X \).

4. Prove the Pasting Lemma: Suppose \( X = A \cup B \), where \( A \) and \( B \) are closed subsets of \( X \). Suppose that \( f : X \rightarrow Y \) is a function, where \( f|_A : A \rightarrow Y \) and \( f|_B : B \rightarrow Y \) are both continuous. Please show that \( f \) is continuous. (Hint: use the strategy suggested in the above question). We will be using the Pasting Lemma a lot. Is this result still true if we do not require that \( A \) and \( B \) be closed?

5. Prove that \( X \) and \( Y \) are compact if and only if \( X \times Y \) is compact. (Hint: I will post a solution to this question on the course website that excuses you from actually writing a solution to this problem. How’s that for a freebie!?)

6. Prove that any closed subset of a compact space is compact.

7. Prove that the following statements are equivalent for a continuous bijection \( f : X \rightarrow Y \):
(a) \( f \) is a homeomorphism.

(b) \( f(U) \) is open in \( Y \) for every open \( U \) in \( X \). (we call \( f \) an open map if this is true.)

(c) \( f(V) \) is closed in \( Y \) for every closed \( V \) in \( X \). (we call \( f \) a closed map if this is true.)

8. Prove that homeomorphism is an equivalence relation. That is, please prove that if \( X, Y \) and \( Z \) are topological spaces:

   (a) \( X \approx X \)
   (b) If \( X \approx Y \) then \( Y \approx X \).
   (c) If \( X \approx Y \) and \( Y \approx Z \), then \( X \approx Z \).

9. Regard \( S^1 \subseteq \mathbb{C} \), so that it inherits the multiplicative structure in \( \mathbb{C} \). Let \( Z_n = \{ \lambda \in S^1 | \lambda^n = 1 \} \) be the \( n^{th} \) roots of unity. Define an equivalence relation \( \sim \) as \( z_1 \sim z_2 \) if and only if \( \frac{z_1}{z_2} \in Z_n \). Let \( X_n = S^1 / \sim \). Prove that \( X_n \approx S^1 \).

10. Suppose \( f : X \rightarrow Y \) is continuous, and that \( X \) is connected. Prove that \( f(X) \) is also connected.