Hi everyone! If anyone knows who I am, I’ll be impressed. I’m a very small and meek reindeer that, for some reason, can change shapes into a huge and menacing reindeer. This seems to contradict the physical law of conservation of mass. Enjoy the solutions!

1. Number 3: We show that the mapping $\varphi(x) = \sqrt{x}$ is an automorphism of the group of positive real numbers under multiplication. This function is operation preserving, as $\varphi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \varphi(x)\varphi(y)$. It’s 1-1 since $\varphi(x) = \sqrt{x} = \sqrt{y} = \varphi(y) \Rightarrow x = y$. It’s onto, for, if $z \in \mathbb{R}^+$, then $z^2 \in \mathbb{R}^+$, and $\varphi(z^2) = \sqrt{z^2} = |z| = z$. So $\varphi$ is an automorphism.

2. Number 4: We observe that $U(8)$ is not cyclic, since it is a group of order 4, and every element has order 2. Next we observe that $U(10)$ is cyclic, as $U(10) = \langle 3 \rangle$. An isomorphism between these groups would preserve the property of being cyclic, and since one is cyclic and one is not, these groups are not isomorphic.

3. Number 10: First we assume that $G$ is abelian, and show that the map $\alpha(x) = x^{-1}$ is an automorphism. It is operation preserving since $\alpha(xy) = (xy)^{-1} = (yx)^{-1} = x^{-1}y^{-1} = \alpha(x)\alpha(y)$. It is 1-1 since $\alpha(x) = x^{-1} = y^{-1} = \alpha(y)$ if and only if $x = y$, and it is onto since $\alpha(y^{-1}) = (y^{-1})^{-1} = y$ for any $y \in G$. So $\alpha$ is an automorphism.

Next we assume that $\alpha$ is an automorphism and show that $G$ is abelian. Let $a, b \in G$. Then on the one hand, $\alpha(ab) = (ab)^{-1}$, but on the other hand, $\alpha(ab) = \alpha(a)\alpha(b) = a^{-1}b^{-1}$. So $(ab)^{-1} = a^{-1}b^{-1}$, and by inverting both sides, we see that $ab = (a^{-1}b^{-1})^{-1} = (b^{-1})^{-1}(a^{-1})^{-1} = ba$. 


4. Number 35: First we point out that Corey seems to remember showing in class that \( \varphi_a \varphi_b = \varphi_{ab} \). If this isn’t the case, then please give Corey a head start to run before calling the people in the white coats. But using \( a = b \) and repeatedly applying this property, we see that \( [\varphi_a]^n = \varphi_{a^n} \) for \( n \geq 2 \), and using \( a^{-1} \) instead of \( a \), and \( b = a^{-1} \) shows that \( [\varphi_a]^n = \varphi_{a^n} \) for \( n \leq 0 \). The case \( n = 1 \) is obvious, and using \( b = e \) establishes the case for \( a^{-1} \) when \( n = -1 \). So \( [\varphi_a]^n = \varphi_{a^n} \) holds for all \( n \). The rest of the proof is straightforward.

Let \( |a| = k \). Then we see that \( [\varphi_a]^k(x) = \varphi_{a^k}(x) = \varphi_e(x) = exe^{-1} = x \) for any \( x \in G \). Thus \( [\varphi_a]^k = \) the identity automorphism. So we conclude that whatever the order of this object is, \( |\varphi_a||k| \).

An example can be seen with \( a = R_{90} \in D_4 \). Notice that \( |a| = 4 \), and that \( a^2 \in Z(D_4) \). So \( [\varphi_a]^2(x) = a^2xa^{-2} = xa^2a^{-2} = x \) for all \( x \in D_4 \). So \( |\varphi_a| \leq 2 \). Since \( \varphi_a \) is not the identity automorphism since \( a \notin Z(D_4) \), \( |\varphi_a| > 1 \). So \( |\varphi_a| = 2 \), and \( 1 < |\varphi_a| < |a| \).