Directions: Please take this exam without cheating. It’s worth 200 points. Please also read the directions to each question carefully—questions that you do not follow the directions on may result in a score of zero points for that question. For this exam, always assume that $V$ is a vector space, and that the dimension of $V$ is finite. Also, $T$ is a linear operator, and $E_\lambda$ and $K_\lambda$ are the eigenspace and generalized eigenspace of $T$ corresponding to the eigenvalue $\lambda$. All other notation will be the notation we’ve developed throughout the course. Anything else which is unclear will be well defined for you. You may use a writing utensil, a calculator, and your brain. Oh, and Rock on!!!

1. (25 points) Please do exactly two of the following.
   
   (a) Let $T$ be a linear operator. Please carefully define what it means for $\lambda$ to be .................... Please also carefully define what it means for .................... And finally, please show that if $\{v_1, \ldots, v_k\}$ are eigenvectors of $T$ corresponding to different eigenvalues (i.e., $v_i \in E_{\lambda_i}$, and $\lambda_i \neq \lambda_j$), then ....................

   (b) Please carefully define the spaces .................... and $K_\lambda$. Please also show that if $0 \neq v_\lambda \in K_\lambda$ and $0 \neq v_\eta \in K_\eta$ where $\lambda \neq \eta$, then the set $\{v_\lambda, v_\eta\}$ is linearly independent.

   (c) Please carefully define..................... Please also show that for any $x, y \in V$ that $T^*(x + y) = T^*(x) + T^*(y)$.

   (d) Please carefully define what it means for $T$ to be a normal operator. Please also ....................

   (e) Please carefully define the Jordan block $J(\lambda, k)$. Please also carefully define ....................

2. (35 points) Please do exactly one of the following.

   (a) For this problem, define $A$ as follows:
   
   Please do all of the following.
   
   i. Please compute all of the .................... Please also ....................
   ii. For each eigenvalue $\lambda$, please compute the dimension ....................
   iii. ....................? If so, please.................... If not, please describe why.

   (b) Suppose $T$ and $U$ are linear operators, $\text{Spec}(T) = \text{Spec}(U)$, and that $|\text{Spec}(T)| = |\text{Spec}(U)| = \dim(V)$. Under these assumptions, please do both of the following.
   
   i. True or False: “....................” If so, please prove that they are. If not, the please provide a counterexample.
   ii. Suppose that $\text{Rank}(T - \lambda I) \neq \text{Rank}(U - \lambda I)$ for some eigenvalue $\lambda$. ....................?

   (c) Suppose $\lambda, \eta \in \text{Spec}(T)$, with .................... Let $\beta = \{v_1, v_2\}$, and let $\gamma = \{w_1, w_2\}$. If $\beta$ and $\gamma$ are linearly independent, ....................

3. (35 points) Please do exactly one of the following.

   (a) Show $T$ is .................... if and only if $V = W_1 \oplus \cdots \oplus W_n$, where....................

   (b) Let $\lambda$ be an eigenvalue of $T$, and let $m$ be the multiplicity of $\lambda$. Please .................... Please give an example where the inequality is strict.

   (c) Let $S$ and $T$ be subspaces of $V$, let $\beta_S$ be a basis for $S$, and let $\beta_T$ be a basis for $T$. Please show that ....................

   (d) Let $W$ be a $T$–invariant subspace of $V$, and let $T_W$ be the operator $T$ restricted to $W$. ....................

4. (35 points) Please do exactly one of the following.
(a) Suppose that $\langle \cdot , \cdot \rangle_1$ and $\langle \cdot , \cdot \rangle_2$ are both inner products for the same vector space $V$. ................................................ (that is, they are positive real numbers), then ................................................

(b) Please do all of the following. Assume that $\langle \cdot , \cdot \rangle$ is an inner product on $V$.
   i. Show that if ....................... for all $y \in V$, then .......................
   ii. Suppose that the set $\{v_1, \ldots, v_k\}$ is an orthogonal set of vectors. Show that it is also ....................... iii. Suppose $T$ is a linear operator on $V$, and that the set $\beta = \{v_1, \ldots, v_n\}$ is an orthonormal basis for $V$. If $A_{ij}$ is the $(i,j)$ entry of $[T]_{\beta}$, then prove .......................

(c) Suppose $\beta$ is an orthonormal basis for $V$. .......................

(d) Let $g$ be any polynomial, and suppose that $W$ is $T$–invariant. .......................  

5. (35 points) Please do exactly one of the following.

(a) Please do all of the following. Assume $V$ is an inner product space.
   i. Suppose $W$ is a subspace of $V$, and that $\beta_W$ is ....................... Show that $x \in W^\perp$ if and only if .......................  
   ii. ....................... Please prove that $||Tx|| = ||T^*x||$ for all $x \in V$.
   iii. Please show that if $T$ is normal, then .......................  

(b) ....................... $K_\eta \subseteq K_\lambda ^\perp$.

(c) Suppose $T$ is an orthogonal operator on a real vector space. Please do both of the following.
   i. .......................  
   ii. .......................  

6. (35 points) Please do exactly one of the following.

(a) Suppose $T$ and $U$ are self-adjoint operators on a real vector space. ....................... there exists an orthonormal basis of eigenvectors of both $T$ and $U$.

(b) .......................  

(c) Suppose $T_1^2 = T_2^2 = 0$, where $T_1$ and $T_2$ are linear operators on $V$. Show that $T_1$ is Jordan equivalent to $T_2$ .......................  

(d) Suppose that $V$ is an inner product space of dimension 3, and that there exists a basis $\beta$ so that $[T]_{\beta} = J(\lambda, 3)$. Show that if ....................... , then $\beta$ is not an orthonormal basis. (Hint: Show that the....................... fails by assuming that ....................... and finding a vector $x$ so that .......................)

7. For 2 free points, tell me something funny!