Here are the solutions to your recent midterm. Enjoy!

(1) (a) (i) \((0, 8]\). (ii) \(\mathbb{R} - \mathbb{Z} = \cup_{n \in \mathbb{Z}} (n, n + 1)\). (iii) Consider \(S_n = (0, \frac{1}{n})\).

(b) This can be found in the book.

(c) Quoting the posted homework 1 solutions: We prove this by double-inclusion. Let \(x \in S \times T\). Then there exists \(s \in S\) and \(t \in T\) with \(x = (s, t)\). Then \(x \in T_s\), and so \(x \in \cup_{x \in S}T_x\). Now let \(x \in \cup_{x \in S}T_x\). Then for some \(s \in S\), \(x \in T_s\), and so there is some \(t\) for which \(x = (s, t)\). Therefore \(x \in S \times T\).

(d) We use double-inclusion. Clearly \(\{2x | x \in \mathbb{R}\} \subseteq \mathbb{R}\). Now let \(y \in \mathbb{R}\). Let \(x = \frac{y}{2}\), so that \(y = 2 \frac{y}{2} = 2x \in \{2x | x \in \mathbb{R}\}\). The latter sets \(\{2n | n \in \mathbb{N}\}\) are not equal, since \(3 \in \mathbb{N}\) but \(3 \neq 2n\) for any natural number \(n\).

(2) (a) Done in class and in the book.

(b) (i) \(f(A) = \{1, 16, 81\}\). (ii) Since the listed domain of \(f\) is \(\mathbb{N}\), \(f^{-1}(B) = \{1, 2, 3\}\). (iii) Yes, \(f\) is injective: if \(a^4 = b^4\), then by taking fourth roots, and since 4 is even, \(a = \pm b\). Since \(a\) and \(b\) are natural numbers (that is, they are both positive), we must have \(a = b\).

(c) This was done in class and in the book.

(d) This was done in class and in the book.

(3) (a) For \(n = 4\), we have \(2^4 = 16 \leq 24 = 4!\), and so the base case holds. Now suppose that for some \(n \geq 4\) we have \(2^n \leq n!\). We have to prove that \(2^{n+1} \leq (n+1)!\). We have

\[
2^{n+1} = 2 \cdot 2^n \leq 2 \cdot n! \leq (n+1) \cdot n! = (n+1)!.
\]

(b) This was done in class, and in the book, but we include a proof here. Clearly, if \(|X| = 1\), then \(X\) contains only one element, and this element must be the largest in the set. Now suppose that any set with cardinality \(n\) contains a largest element. Let \(X\) be a set with \(|X| = n + 1\). Choose any \(x \in X\), and consider the set \(X - \{x\}\). \(|X - \{x\}| = n\), and so there is a largest element \(y \in X - \{x\}\). The set \(X\) is the disjoint union of \(X - \{x\}\) and \(\{x\}\). Let \(z\) be the larger number between \(x\) and \(y\). Then \(z\) is larger than every element of \(X\) since \(z \geq y\) and \(y\) is larger than every element of \(X - \{x\}\), and \(z \geq x\) by construction. So \(z\) is the largest element of \(X\).

(c) This was proven in class and is in the book.

(d) Using 1(c) above, \(S \times T\) is a union of sets. This union is indexed by \(S\), so the union is a countable one. Each set \(T_x \sim T\) via the bijection \((x, t) \mapsto t\). Since \(T\) is countable, so must \(T_x\) be countable, and so each set being unioned is countable. Thus, \(S \times T\) is the countable union of countable sets, and is therefore countable by one of our results.

(e) We know that \(\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})\), where the latter set is the irrational numbers. Suppose by way of contradiction that the irrational numbers were countable. Since
Q is countable, we would have $\mathbb{R}$ as the union of two countable sets, and as the countable (finite) union of countable sets, it would follow that $\mathbb{R}$ is countable. This is a contradiction, since $\mathbb{R}$ is uncountable.

(f) This was done in class and in the book.

(g) We prove this by induction. Clearly, if we cross product only one copy of $\mathbb{N}$, this is a countable set. Now suppose that the cross product of $n$ copies of $\mathbb{N}$ is countable. The cross product of $n+1$ copies of $\mathbb{N}$ is the cross product of $\mathbb{N}$ with the cross product of $n$ copies of $\mathbb{N}$, and both of these are known to be countable: the first since $\mathbb{N}$ is countable, and the second by the induction hypothesis. According to 3(d), such a cross product is again countable, proving the claim. (To answer the extra credit problem, such a countably infinite cross product of $\mathbb{N}$’s is uncountable: it contains the countably infinite cross product of $\{1, \ldots, 9, 10\}$, and there is an onto map from this cross product onto $[0,1]$ by expressing each sequence of numbers as the decimal expansion of a number in $[0,1]$. Moreover, every number in $[0,1]$ has such an expansion, so this association is indeed an onto one. Since $[0,1]$ is uncountable, this subset is uncountable, hence the original cross product is uncountable.)

(h) Set

$$f(x) = \begin{cases} 
  x & \text{if } x \neq 2^{-n+1} \text{ for } n \in \mathbb{N} \\
  2^{-n+2} & \text{if } x = 2^{-n+1}
\end{cases}.$$

This is clearly 1-1, since the only thing this function changes is an advance of the sequence $2^{-n+1}$ (similar to a proof of ours), and is onto, since every one of the outputs is in $(0,1)$: if $x$ is not one of $2^{-n+1}$, then $x$ is mapped from itself. If $x$ is one of the $2^{-n+1}$, then it is mapped to by the previous one in the sequence. The only exception to this is that when $n = 1$, this does not have a predecessor to be mapped from, and this is the output $2^{-1+1} = 2^0 = 1$, and $1 \not\in (0,1)$.

(i) Let $f(x) = \frac{1}{x} - 1$. Now if $x \in (0,1)$, then $0 < x < 1$, so $1 < \frac{1}{x}$, so $0 < \frac{1}{x} - 1$, so this function actually sends inputs in $(0,1)$ to outputs in $(0,\infty)$. This function is 1-1, since if $f(x_1) = f(x_2)$, then $\frac{1}{x_1} - 1 = \frac{1}{x_2} - 1$, so $x_1 = x_2$. This function is onto, since if $y \in (0,\infty)$ is given, then solving for $x$ in $\frac{1}{x} - 1 = y$, we have $x = \frac{1}{y+1}$, so that $f(x) = y$. 
