Hi everyone, please enjoy these selected solutions from your HW # 4.

2.2.1 (a) sup\((S)\) = 4, inf\((S)\) = −7. (b) sup\((S)\) = ∞, inf\((S)\) = 0, (c) sup\((S)\) = ∞, inf\((S)\) = 1, (d) sup\((S)\) = ∞, inf\((S)\) = −∞.

2.2.4a For ease of notation, denote \(\alpha = \text{sup}(S)\). We want to show that \(a\alpha = \text{sup}(aS)\), and we do that by showing that \(a\alpha\) is (1) an upper bound, and (2) the least upper bound.

(1) Let \(as \in aS\), for \(s \in S\). Since \(s \in S\) and \(\alpha = \text{sup}(S)\) is an upper bound for \(S\), we must have \(s \leq \alpha\). Since \(a > 0\), we must then have \(as \leq a\alpha\). This shows that \(a\alpha\) is an upper bound for \(aS\).

(2) Let \(y < a\alpha\). Then since \(a > 0\), \(\frac{y}{a} < \alpha\). Since \(\alpha\) is the least upper bound of \(S\), there must exist an \(s \in S\) with \(\frac{y}{a} < s\), so \(y < as\). This shows that the number \(y\) is not an upper bound for \(aS\).

2.3.1ac (a) This infimum is 0. Clearly, (1) 0 is a lower bound for this set. (2) If we choose any number greater than 0, say, \(\epsilon > 0\), then the Archimedean Property says there exists an \(N \in \mathbb{N}\) with \(\frac{1}{N} < \epsilon\). Notice that \(\frac{1}{N} \in \{\frac{1}{n} | n \in \mathbb{N}\}\), so \(\epsilon\) fails to be a lower bound for the set.

(c) This supremum is 1. For easy reference, denote \(S = \{r \in \mathbb{Q} | 0 < r < 1\}\). To see that the supremum is 1: (1) 1 is clearly an upper bound for this set. (2) Choose any \(y < 1\). It is either the case that \(y \leq 0\) or \(0 < y < 1\). If \(y \leq 0\), then the rational number \(\frac{1}{2} \in S\) satisfies \(y < \frac{1}{2} < 1\), and so \(y\) fails to be an upper bound in this case. If \(0 < y < 1\), then since the rational numbers are dense in the real numbers, there exists an \(r \in \mathbb{Q}\) so that \(y < r < 1\).

2.3.8 Suppose there were only finitely many rational numbers between distinct real numbers \(x\) and \(y\) (with \(x < y\)). Denote these finitely many rational numbers as \(r_1, \ldots, r_n\), with \(x < r_1 < r_2 < \cdots < r_n < y\).

Since the rationals are dense in the reals, there exists another rational number \(r\) with \(x < r < r_1\).

This is a contradiction: the only rational numbers between \(x\) and \(y\) were \(r_1, \ldots, r_n\), and the additional rational number \(r\) cannot equal any of those since it is strictly smaller than all of them.