Hello everyone. Please enjoy these selected solutions of HW 3.

1.5.1 Since \( S \) is countable, it is either countably infinite or finite. In the former case (\( S \) is countably infinite), by definition, \( S \sim \mathbb{N} \), and thus, there is a bijection (hence surjection) from \( \mathbb{N} \) to \( S \). In the latter case (\( S \) is finite), then we can express \( S = \{ s_1, \ldots, s_k \} \). Define the function \( f : \mathbb{N} \rightarrow S \) by the following rules:

\[
f(1) = s_1, \ldots, f(k) = s_k, \text{ and } f(x) = s_k \text{ for } x \geq k + 1.
\]

This is a surjection from \( \mathbb{N} \) to \( S \).

1.5.3 The following function is a bijection from \((0, 1]\) to \((0, 1)\):

\[
f(x) = \begin{cases} 
\frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\
\frac{x}{n} & \text{otherwise}.
\end{cases}
\]

1.5.8 We express \( S \times T \) as the countable union of countable sets. First, for \( x \in S \), recall the set

\[
T_x = \{ (x, t) | t \in T \}.
\]

The bijection \( t \mapsto (x, t) \) shows that \( T \sim T_x \), and so since \( T \) is countable, so it \( T_x \). We also showed in an earlier homework problem that \( S \times T = \bigcup_{x \in S} T_x \). Here, the indexing set \( S \) is countable, and so \( S \times T \) is the countable union of countable sets, which by one of our theorems, is countable.

1.5.9 Suppose the irrational numbers were countable. We know that \( \mathbb{Q} \) is countable and \( \mathbb{R} \) is uncountable. And, of course, that \( \mathbb{R} = \mathbb{Q} \cup \) the irrationals. This is a problem if the irrationals are countable: we would find that \( \mathbb{R} \) is the union of two countable sets, which is the countable union of countable sets, which would have to be countable by a theorem of ours—a contradiction.

2.1.9 (a) Following the hint, replacing \( b = c - a \) we have

\[
|a + b| \leq |a| + |b| \iff |a + (c - a)| = |c| \leq |a| + |c - a|, \text{ so } |c| - |a| \leq |c - a|.
\]

(b) Following the hint, replacing \( \tilde{b} = a - c \) we have

\[
|c + \tilde{b}| \leq |c| + |\tilde{b}| \iff |c + (a - c)| = |a| \leq |c| + |a - c|, \text{ so } |a| - |c| \leq |a - c| = |c - a|.
\]

(c) This follows directly from the previous parts.