The Twisted Torus And Knots

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Abstract

This paper looks at the construction of a twisted torus and it proves that when putting a $T_{r,s}$ around a twisted torus a new torus knot $T_{r,z}$, results where z is a function of s, r, and n, where n is the number of twists in the torus. This paper shows that the stick representation on a twisted torus is $2ns + 2s$ which improves the known upper bounds for the stick number as given in Jin's construction.

1 Introduction

A torus is a surface with a hole in it. A real life example of a torus is a doughnut. On the surface of the torus, meridian and longitude curves can be drawn. The meridian curve is a curve that goes around the shorter way of the torus, while the longitude curve is drawn so that it runs the long way around the torus.

A torus knot is a knot on the surface of the torus and it can be characterized by the number of times it crosses the meridian and longitude of a torus. For example, if a knot crosses the longitude three times and the meridian two times, it would be characterized as a $T_{2,3}$ knot, with the general form of $T_{r,s}$ where r represents the number of times the knot crosses the meridian and s represents the number of times the knot crosses the longitude.

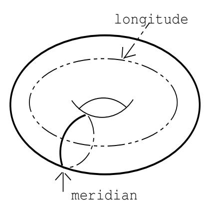

Figure 1: The torus with its meridian and longitude drawn in.

2 Minimal Stick Number

Jin looks at the minimal stick number for knots on a torus. The stick number $s(T_{r,s})$ of a knot is the least number of sticks needed to make a knot representation. Jin proves if $2 \leq r < s < 2r$, then $s(T_{r,s}) = 2s$. Furthermore, Jin proves for r$\geq$ 1, $s(T_{r,r}) = 3r$ and $s(T_{r,2r}) = 4r - 1$.

2.1 Jin's Construction of the Torus

In order to prove the minimum stick number, Jin constructs the torus out of hyperboloids. Jin begins by choosing an angle, α, between $\pi r/s$ and min $\{\pi, 2\pi r/s\}$. Then, he chooses two points $(cos(\alpha/2), -sin(\alpha/2), -1)$ and $(cos(\alpha/2), sin(\alpha/2), 1)$ from which he constructs a line segment.

The line segment Jin constructs has equations $x = cos(\alpha/2)$ and $y = zsin(\alpha/2)$. Jin rotates this line segment around the z-axis to obtain the hyperboloid with the equation $x^2 + y^2 - z^2 sin^2(\alpha/2) = cos^2(\alpha/2)$. Similarly Jin takes another angle, β, (defined to be β =2πr/s-α) and using the same procedure as above forms another hyperboloid.

Jin takes the union of the two hyperboloids and the result is a torus with the alpha hyperboloid (H_α) on the outside and the beta hyperboloid (H_β) of the inside. Figure 2 shows the completed torus based on Jin's construction.

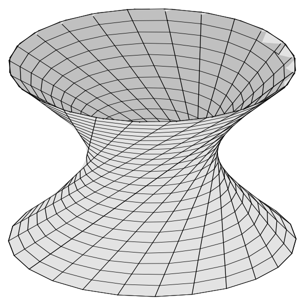

Figure 2: The union of the hyperboloids

2.2 Jin's Construction of Knots on the Torus

In the next step of the process, Jin places the knot onto the torus by constructing line segments which rotate to form the torus knot. Jin constructs the line segments from the points below, where i goes from 0 to $2s$.

$$X_i = \begin{cases} (cos(\pi(r)i/s), sin(\pi(r)i/s), -1) & \text{if } i \text{ is even;} \\ (cos(\pi(r)(i-1)/s + \alpha), sin(\pi(r)(i-1)/s + \alpha), 1) & \text{if } i \text{ is odd.} \end{cases}$$

By having an even i, $\overline{X_i X_i + _1}$ is rotated through the angle, $-\pi ri/s - (\alpha/2)$ on the z-axis. This rotation causes the points $(cos(\alpha/2), sin(\alpha/2), 1)$ and $(cos(\alpha/2), sin(\alpha/2), -1)$ to be connected by the line segment. This line segment is contained in H_α. The same technique can be applied to the line segment $\overline{X_i X_{i-1}}$, while rotating it through the angle, $-\pi ri/s - (\beta/2)$ (radians) on the z-axis. This line segment is contained in H_β.

Each curve goes around the $z - axis$ through $2\pi r/s$ (radians), so a knot goes r times around the longitude. Then Jin considers a disc whose core is the union of H_α and H_β. The curve that Jin defines causes the linking number of the knot to be s, meaning that he has constructed $T_{r,s}$ with $2s$ edges.

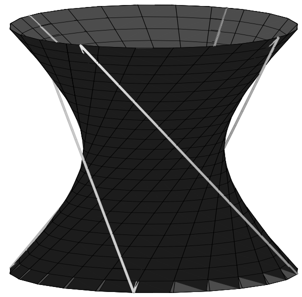

Figure 3: The $T_{3,4}$ knot on the torus using Jin's construction

3 Linking, Twisting, and Writhing

In the 1980's, biologists were working with DNA and they discovered DNA had similar properties to knots. The DNA was supercoiled which relates to the twisted torus and its behavior. Supercoiling can be defined as the difference between the linking number of DNA in its natural state and the linking number of a molecule in a different closed state. (The closed state would be where the writhing is equal to zero and the natural state of DNA is when writhing is at its greatest.) The supercoiling configuration has the minimum amount of twisting and it introduces bend into the DNA molecule. Supercoiling increases the writhing in the DNA molecule.

Writhe is equal to the linking number minus the twisting number. The twist of a ribbon measures how much a ribbon twists around its axis and it is defined to be the integral of the incremental twists around the ribbon.

When a knot crosses over itself, it produces an index number of 1 or -1, depending on the way the top piece crosses over the bottom piece. The linking number is half the sum of the signs of the crossings. The positive crossing has the right strand crossing under the left strand, while the negative crossing has the right strand crossing under the left strand. Figure four depicts the orientation of the knot and the type of crossing it produces.

One way to think about writhing, linking, and twisting would be to imagine a telephone cord. When a telephone cord is in its natural state its twisting is small and its writhe is large. However, when the telphone cord is stretched, the the writhing on the telphone cord becomes small and the twisting increases.

Linking is important when constructing the knot on the twisted torus. The linking number of a knot on a torus can be increased by writhing and twisting. Jin's construction uses twisting to increase the linking number. However, when placing a knot on the twisted torus the linking number increases due to writhing and twisting.

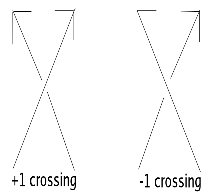

Figure 4: Positive and negative crossings.

4 Constructing the Twisted Torus

Based on the ideas of the supercoiled DNA properties and the Jin's construction of the hyperboloid, a twisted torus was created by adding twists to the torus. By twisting the torus, writhe was added.

The single twisted torus is made up of an upper and a lower twisted loop that form the boundaries for the twisted hyperboloid. In this construction, one full loop can be made from two equations.

The points in A form the bottom loop of the twisted torus.

$$A = \begin{cases} (cos(\alpha), sin(\alpha), 1 - .35cos(\alpha/2)) & \text{if } 0 \leq \alpha \leq 2\pi \ ; \\ (2cos(\alpha) - 1, 2sin(\alpha), 1 - .35cos(\alpha/2)) & \text{if } 2\pi \leq \alpha \leq 4\pi. \end{cases}$$

The points in B form the top loop of the twisted torus.

$$B = \begin{cases} (cos(\alpha), sin(\alpha), 1.5 - .35cos(\alpha/2)) & \text{if } 0 \leq \alpha \leq 2\pi \ ; \\ (2cos(\alpha) - 1, 2sin(\alpha), 1.5 - .35cos(\alpha/2)) & \text{if } 2\pi \leq \alpha \leq 4\pi. \end{cases}$$

When the two loops are graphed together loop B sits above loop A and a link is formed between the two.

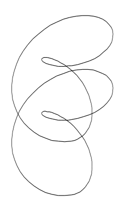

Figure 5: The bottom and top of the twisted hyperboloid.

The figure of the twisted torus can be thought of having two main parts. The "smooth" part is the part of the torus that connects from the bottom to the top of the torus. This is the part that does not have twisting. The "twisted" part of the torus is obviously the part of the torus that twists. The twisted part of the torus is also the part where the linking occurs.

From the construction the twisted hyperboloid is the surface between the top and lower boundaries of the loop. As in Jin's construction, the twisted torus has an inside and outside edge so the knot can alternate.

5 Constructing the Twisted Torus Knot

In order to graph the knot onto the twisted torus, Jin's scheme to plot the points onto the hyperboloid was used. To graph the points onto the twisted torus for i from 0 to $4s-1$, the following construction was used, where b=πri/s, d=$b/2$ and j=πr(i-1)/s +((r+.5)π/s). This will hold for when $0\leq b<2\pi$ or $4\pi \leq b<6\pi$ or $8\pi \leq b<10\pi$.

If k is even, then the following formulas were used:

$$X_k = \begin{cases} (cos(b), sin(b), 1 - .35cos(d)) & \text{if } i \text{ is even;} \\ (2cos(b) - 1, 2sin(b), 1 - .35cos(d)) & \text{if } i \text{ is odd.} \end{cases}$$

If k is odd, then the following formulas were used.

$$X_k = \begin{cases} (cos(j), sin(j), 1.5 - .35cos(j/2)) & \text{if } i \text{ is even;} \\ ((2cos(j) - 1, 2sin(j), 1.5 - .35cos(j/2)) & \text{if } i \text{ is odd.} \end{cases}$$

The different formulas were used when k was odd and even is because Jin's construction has a different pattern for even and odd cases. For example in Jin's construction he skips two vertices for a $T_{3,s}$ and for a $T_{4,s}$ he skips first two vertices and then he skips four vertices to achieve the proper arrangement of the knot. This knot construction will allow for the knot to alternate from the top loop of the torus to the bottom loop of the torus. Furthermore, the knot will alternate from the inside to the outside of the torus. The vertices are spaced evenly apart.

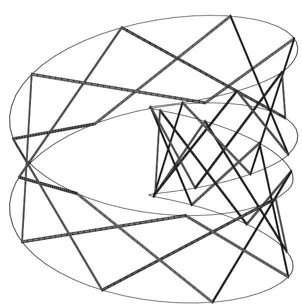

Figure 6: The knot with base $T_{3,8}$ on a single twisted torus.

6 The Representation of Knots on the Twisted Torus

To draw Jin's scheme on a piece of paper, a circle is constructed with twice r vertices. For $T_{3,s}$ a line would connect the vertices that are spaced two apart, as seen in the diagram.

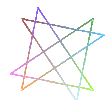

Figure 7: The representation of a $T_{3,4}$ on a torus using Jin's construction

For the twisted torus a different paper representation has to be used to graph the knot onto the torus. The twisted torus can be made so it had two circles as shown in the figure. In a diagram with one twist, one set of vertices would be placed on the inside circle and other set of vertices from the other knot would be placed on the outside circle. The knot would be drawn in the same manner as the knot on the torus; however, the knot would have to alternate from the inside to the outside circles.

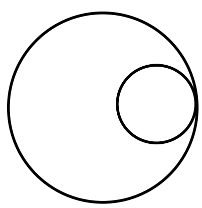

Figure 8: The twisted torus drawn two dimensional.

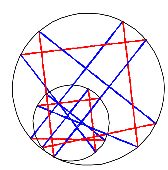

Figure 9: The representation of $T_{3,4}$ on the twisted torus.

7 $T_{3,s}$ on the Twisted Torus

In his work, Jin gives an example of the knot $T_{3,4}$ using eight points with each line segment alternating on the inside and outside the hyperboloid. Jin's knot construction used eight sticks to go around the hyperboloid.

Using the twisted torus, the knot was plotted on it with the base knot $T_{3,4}$. The knot was carefully examined and each of the line segments went inside and outside of the twisted torus as in Jin's construction. The knot on the single twisted torus was at least $T_{3,8}$ because one $T_{3,4}$ knot was added to the "twisted" part of the torus and one $T_{3,4}$ was added to the "smooth" part of the torus.

However, since the two boundaries for the twisted torus link, the linking number had to be considered. The way the twisted torus was constructed six strands of the knot crossed over the twisted part with each having a positive crossing of one. The linking number is three because six strands cross over the core of the torus contributing a net of three. This is demonstrated in figure 7. The linking number of three was contributed to the knot, thus making the knot $T_{3,11}$. The stick number of $T_{3,11}$ is sixteen which results from twice the number of vertices.

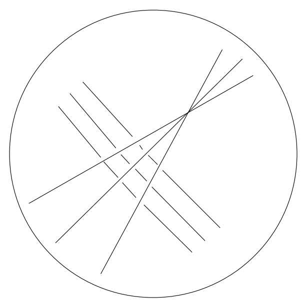

Figure 10: The linking with the core for a $T_{3,s}$

By extending the following procedure with each of the $T_{3,s}$ knots, a formula was made as to what knot would result from the base knot $T_{3,s}$, where is s is relatively prime to three. All the knots of general form $T_{3,s}$ have a linking number of three. Furthermore, as the knot acquires more twists, the number of base knots have to be added for each twist of the torus plus one base knot for the "smooth" part of the torus.

Lemma 7.1 *The knot $T_{3,z}$ where $z = (s(n+1)+3n)$, where s is from the base knot and n is the number of twists in the torus.*

Lemma 7.2 *The stick number for the knot $T_{3,z}$ is $2(s(n+1))$ which is twice the number of all vertices of the base knots on the torus.*

8 $T_{r,s}$ on the Twisted Torus

Extending the idea of the base $T_{3,s}$ knots on the twisted torus, torus knots of the general form $T_{r,s}$ were examined. It was observed from $T_{3,s}$ that the linking number has to be considered because the torus twists. In general, the linking number will be the number, r, of the base torus knot. The linking number will be r because this is the half number of times the knot goes along the longitude of the torus. Based on this construction, a general formula can be constructed that determines the knot that results from the base knot.

As in the $T_{3,s}$ case, the torus can be twisted more than one time so the base knot has to be added the number of times the torus is twisted. In general for a base $T_{r,s}$ knot on a twisted torus, the number of times the torus twists the more times the base knot has to be added.

Lemma 8.1 *The new $T_{r,z}$ that will result is $z = s(n+1) + rn$, where r and s are the meridian and longitude crossings in the base knot and n is the number of twists the torus has.*

The stick number for $T_{r,s}$ is $2(s(n+1))$. Note this is the same as the $T_{3,s}$ base knot case because the stick number does not depend on the linking number. The stick number depends on the number of times the base knot has been added around the twisted torus.

9 Proof

A $T_{r,s}$ knot intersects the meridian r times and the longitude s times, while the meridian and longitude intersect exactly once. (The number of intersections with the meridian is the longitude number.) In general, $T_{r,s}$ $=s\vec{m}+r\vec{l}$, so this concept can be applied to the twisted torus knots to show that they are torus knots.

In general for $T_{p,q}$, the torus knot is equal to $q\vec{m}+p\vec{l}$, where $\vec{m}$ is the number of crossings of the meridian and $\vec{l}$ is the number of crossings of the longitude. It is also the unique disc that bounds the disc outside of the torus. However to see if the knot that was constructed by adding twists is a torus knot, p and q have to be found in terms of r, s and, n.

It is known that $T_{p,q}$= $A\vec{m}+B\vec{l'}$ where $\vec{l'}$ interesects with the top of the torus and A and B are constants. To find $\vec{l'}$ the intersections of the knot with the meridian need to be calculated. Furthermore, $\vec{l'}$=$A\vec{m}+\vec{l}$. Using these facts, it can be shown that the knots constructed are torus knots. $T_{p,q}$=$A\vec{m}+B\vec{l'}$
=($T_{p,q}$ $\vec{l'}$)$\vec{m}$ + ($T_{p,q}\vec{m}$) $\vec{l'}$, where $T_{p,q}$ $\vec{l'}$ is the number of vertices.
By substituting the values into the equation the following results- $=ns\vec{m}+r\vec{l'}$
however, the equation has to be put in terms of $\vec{l}$ so by substitution
$=(ns)\vec{m}+ r(\mathrm{A}\vec{m}+\vec{l})$
$=(ns)+(rA\vec{m})+ r\vec{l}$
$=(ns + rA)\vec{m}+ r\vec{l}$

Using the equation to give the new torus knot, $(ns + rA)$ gives the number of meridians on the torus knot, while r gives the number of longitudes. It is known that $z = s(n + 1) + rn$ and this is equivalent to the equation above.

10 The Best Construction of $T_{r,z}$ with the Same Base

From Jin's construction, it is known that the stick number is $2s$. However based on the construction of the twisted torus, it is known that the stick number is $2ns + 2s$. For the general case this is $2rn$ sticks less then in Jin's case.

Furthermore, some knots can be made with different bases and different number of twists, yet still result in the same $T_{r,z}$ knot. When observing these knots, the knots with the smaller base knot and more twists used less number of sticks then the ones with the larger base and less twists.

As known from the previous equation, for a $T_{3,s}$ the equation to find the z is $(s + 3)n + s$. To minimize the stick number of s the derivative has to be taken. By solving the z equation for n and substituting it into the stick number equation of $2ns + 2s$ the minimum can be found. The equation for z in $T_{3,s}$ is $z = (s + 3)n + s$ and solving it for n, $n = \frac{z-s}{s+3}$. By substituting it into the minimum stick equation the following equation is the result: $\frac{2zs-s^2}{s+3} + 2s$.

Taking the derivative with respect to s shows that the stick number decreases with more twists and a smaller base knot. Optimally, for $T_{3,z}$ the smallest base knot is needed with the most number of twists. In the $T_{3,s}$ case, the base knot that would have the minimun stick number would be $T_{3,4}$ with n number of twists. S is the smallest it can be and the stick number goes down (in relation to an untwisted tours) once the twists are added to the knot. The same applies to the generalized case where $z = (s + r)n + s$. The stick number would be minimized when n is the greatest and s is the least.

11 Different Bases

Torus knots can be made out of different knots that have the same r. For example, a $T_{3,5}$ and $T_{3,8}$ can be put together to form $T_{3,16}$ with one twist.

Lemma 11.1 *For two knots to be put together, $T_{r,s}$ and $T_{r,t}$, then $T_{r,z}$ would be $z = t + ns + rn$, where s is smaller of the two.*

When constructing a knot out of different bases, sometimes a link forms. When this occurs, the knot would connect back to its starting point after one rotation. To finish the link that is formed, start one vertex to the right of the original starting point and continuing doing this until all the vertices are connected. This will depend on the r of the knot.

Even though different bases are used for the construction of the knot, the stick number remains $2rn$ less then in Jin's construction. The total stick number for a knot with different bases is *2(t+ns+rn)* in Jin's construction. However, using the twisted torus construction the stick number is *2(t+sn)*.

12 Comparison of Stick Number on Different Bases

Based on experimental evidence, it appears that the most efficient torus knot in terms of stick number is the knot with different bases and the most number of twists. For example, $T_{3,67}$ can be made with 92 sticks using one $T_{3,11}$ and seven $T_{3,5}$ twisted seven times. However, $T_{3,67}$ can be made with a base $T_{3,7}$ that is twisted six times and has 98 sticks. With the $T_{3,7}$ base, the knots are all equal (they have the same number of sticks). However, in the base $T_{3,5}$ and $T_{3,11}$, the $T_{3,11}$ adds 11 vertices but when the $T_{3,5}$ is added multiple times it contributes less sticks then $T_{3,7}$. Thus when multiple twists are added to the torus, the stick number is less then the knot with base $T_{3,7}$.

Even though the lower cases of the different bases knots do not appear to give less of a stick number, (they are either more or equal to the number given by the same base knots) as the number of twists increases, then the stick number increases. This is due to the fact that the knot being twisted is smaller than the other knot it is connected to.

13 Conclusions and Recommendations

Using the twisted torus construction, the equations for z of $T_{r,z}$ were found for knots constructed from the same and different bases. It was shown that the stick number of any $T_{r,z}$ was $2sn + 2s$ which is $2rn$ less than Jin's construction. Moreover, it was shown that the stick number is improved with more twisting and a smaller base. Last, evidence suggests that the best stick construction for a torus knot might be with different bases.

Some further areas of investigation might be to prove which bases give the minimum stick number and investigate how the different bases interact with one another. In addition, more research needs to be done on what happens when two or more knots combine together on the twisted torus. The formula for the new knot that results from this combination and its stick number have to be looked at in relation to the other knots construsted. Moreover, a method could be found to determine the most efficient base knots used to create a knot when given z.

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