MATH 614-01: Studies in Geometry (42673)
Syllabus: Spring 2011
MW 6-7:50PM, JB - 383

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Text: Brannan/Esplen/Gray
Geometry (Cambridge University Press)

Prerequisites: MATH 529 and MATH 545

The course objective is to study hyperbolic geometry, commonly called non-Euclidean geometry, as a system of plane geometry independent of but closely related to Euclidean plane geometry. Since the formal study of this geometry emerged in the nineteenth century two fundamental models have been used to formulate its theorems, the inversive and the projective. As with Euclidean geometry, hyperbolic geometry can be developed as a sub-geometry of projective geometry, but we will take the inversive approach because it is easier to capture the conformal (angle-preserving) aspects of the theory and because we can enter the subject directly with a minimum of linear algebra. However, we will begin by establishing the correspondence between the inversive and projective models and by introducing some modern notation that will allow us to compute efficiently.

Grading will be based on **two assignments, an exam on the introductory material, and the final exam**. These four components will be weighted equally. Guidelines for the two assignments will be provided later, but one of these should be configured for your portfolio (both must be typed, preferably in Latex). After computing your total scores weighted according to these percentages, course grades will be assigned as follows:

$$\begin{array}{ccc}
 & A :\geq 91 & A- : 86-90 \\
B+ : 81-85 & B : 76-80 & B- : 71-75 \\
C+ : 66-70 & C : 61-65 & C- : 51-60 \\
D : 45-55 & F :< 45 &
\end{array}$$

Some important dates:

2011-04-04: First day of class; Winter grades available on-line, add open classes (MyCoyote)
2011-04-22: Late Drop period ends
2011-04-08: Last day to add open classes on-line (MyCoyote)
2011-04-11 - - 04/15/2011: Open classes require signatures
2011-04-22: Census Deadline; last day to submit adds/drops
2011-04-27: Exam on introductory material
2011-05-02: Grad Check deadline for Winter (March) 2012
2011-05-16: Fall Advising begins
2011-06-13: Last day of class
2011-06-15: Final Exam

Introductory material:

> It had for a long time been evident to me that geometry can in no way be viewed, like arithmetic or combination theory, as a branch of mathematics; instead geometry relates to something already given in nature, namely, space.
>
> Hermann Grassmann
> *Die Ausdehnungslehre*, 1844

By the middle of the nineteenth century the reality of non-Euclidean geometry had been accepted but with many reservations from the philosophical and scientific communities. This so-called *Euclidean bias* was defended by Immanuel Kant as being innate to human reasoning. However, continuing developments in mathematics and physics led to a new philosophical perspective known as *conventionalism*, which held that all consistent systems of geometry should be of use in describing physical space; in other words, the preference for one system of geometry over another is a matter of convention. The most prominent proponents of conventionalism were Riemann, Helmholtz and Poincaré. Their influence enabled the work of Bolyai, Lobachevsky and earlier work of Gauss on hyperbolic geometry to proliferate (Gauss had never published his work on non-Euclidean geometry, sensing that it would not be readily accepted), and differential geometers such as Eugenio Beltrami established the importance of this system of geometry. Near the end of the nineteenth century mathematics and physics used group theory to standardize the representation of transformations. This revolutionized both subjects. The work of Lorentz led to Einstein's formulation of relativity, and the Erlangen program of Felix Klein defined geometry as a system of incidence preserved by a group of transformations. This era of the *transformational approach* forced renewed attention on the axiomatic foundations of geometry. Early in the twentieth century Hilbert, Pasch and others developed a consistent system of axioms for *absolute* geometry that encompassed the theorems common to Euclidean and hyperbolic geometry, and complete systems for both geometries soon followed. Today, the transformational and axiomatic approaches have merged, through combinatorics and the theory of groups of Lie type, to produce a far-ranging theory of incidence geometries with generalized metric properties. Our study of non-Euclidean geometry will rely on the transformational approach, with the hyperbolic plane represented as a sub-geometry of the inversive plane. An excellent reference for inversive geometry is Hans Schwerdtfeger, *Geometry of Complex Numbers*, Dover 1979. The following outline summarizes in modern notation the main ideas we will need.

I. Representation of circles by Hermitian matrices.

1) Let $z = x + iy$ and let $Z = \left[\begin{array}{cc} z & 1 \end{array} \right]$. If A, B, C, D are real constants and $H = \left(\begin{array}{cc} A & B + iC \\ B - iC & D \end{array} \right)$ then the equation

$$ZHZ^* = 0$$

describes an inversive line in $\widehat{\mathbb{C}}$ provided $\det H < 0$. Since an equivalent equation is obtained from λH when $\lambda \in \mathbb{R}$ is non-zero we view H in this context as a *real projective* matrix (much as we view the equation $ax + by + ct = d$ for a plane in $\mathbb{R}^3$ as a real projective equation), whereby the zero matrix has no representational value. Thus there is a one-to-one correspondence between inversive lines and the real projective Hermitian matrices of negative determinant. This correspondence is consistent with stereographic projection between circles on S^2 (secant planes) and inversive lines in the plane.

2) If $\det H = 0$ then the equation describes a point in $\widehat{\mathbb{C}}$, a circle of radius 0, where

$$H = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$$

represents the point ∞. Now there is a one-to-one correspondence between the points of the completed plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the real projective Hermitian matrices of determinant 0, consistent with stereographic projection between points on S^2 (tangent planes) and $\widehat{\mathbb{C}}$. (A correspondence between the real projective Hermitian matrices of positive determinant and the planes exterior to S^2 consistent with the idea of polarity can also be established.)

II. Inner product.

1) Just as the study of Euclidean geometry is simplified by the introduction of the "dot" product the study of inversive geometry is simplified by defining a (symmetric) inner product on Hermitian matrices:

$$\langle H_1, H_2 \rangle = \operatorname{tr} \left(H_1 H_2^{adj} \right)$$

where H^{adj} is the adjoint matrix of H; thus HH^{adj} is the identity matrix multiplied by $\det H$. It follows that $\langle H, H \rangle = 2 \det H$ and so the determinant of a Hermitian matrix is the analog of the squared length of a Euclidean vector. In case H represents an inversive line we have $\det H < 0$ and so $\sqrt{\langle H, H \rangle}$ is a real multiple of i.

2) As with a Euclidean inner product we can use this inner product to define

$$\cos \alpha = \frac{\langle H_1, H_2 \rangle}{|H_1| \, |H_2|}$$

where $|H| = \sqrt{\langle H, H \rangle}$. When H_1 and H_2 are intersecting inversive lines then $|\cos \alpha| \leq 1$ and α will be the angle of intersection interpreted relative to the orientations of the inversive lines.

III. Pencils.

1) If H_1 and H_2 are not equivalent then $\{\lambda_1 H_1 + \lambda_2 H_2 \mid (0,0) \neq (\lambda_1, \lambda_2) \in \mathbb{R}^2\}$ is called the *pencil* generated by them when viewed as a projective space. Note that the pencil can also be written $\{H_2\} \cup \{H_1 + \lambda H_2 \mid \lambda \in \mathbb{R}\}$ due to the projective equivalence on members of the pencil.

2) We are primarily concerned with pencils generated by two inversive lines. These are classified by the discriminant of the quadratic form $Q(\lambda_1, \lambda_2) = \langle H_1, H_1\rangle \lambda_1^2 + 2\langle H_1, H_2\rangle \lambda_1\lambda_2 + \langle H_2, H_2\rangle \lambda_2^2$, which is the determinant of the matrix

$$Q = \begin{pmatrix} \langle H_1, H_1\rangle & \langle H_1, H_2\rangle \\ \langle H_1, H_2\rangle & \langle H_2, H_2\rangle \end{pmatrix}.$$

Since $\langle H_1, H_1\rangle < 0$ the form will be **negative definite** provided $\det Q > 0$, **negative semi-definite** provided $\det Q = 0$ and **indefinite** provided $\det Q < 0$.

Elliptic pencils: $\det Q > 0$

Then $\det(\lambda_1 H_1 + \lambda_2 H_2) < 0$ and so every member of the pencil represents an inversive line. The pencil consists of all inversive lines that pass through two distinct points, and if ∞ is not one of them then the pencil contains a unique line through ∞.

Parabolic pencils: $\det Q = 0$

Then $\det(\lambda_1 H_1 + \lambda_2 H_2) < 0$ except for one member of the pencil for which the determinant is 0. If this point circle is not ∞, then there is a unique line through ∞ and the pencil consists of all circles tangent to this line at the unique point circle.

Hyperbolic pencils: $\det Q < 0$

Then $\det(\lambda_1 H_1 + \lambda_2 H_2)$ can assume any value. The pencil includes two distinct point circles, and contains members that do not represent inversive lines or points. If ∞ is not one of the point circles then there is a unique line through ∞. In any case, no two inversive lines of the pencil intersect. The inversive lines in the pencil are sometimes called an *Apollonian family of circles*.

IV. The group $\mathcal{M}$ of inversive transformations.

1) By a theorem of Möbius and Riemann, the only conformal transformations of S^2 are those induced by inversion. This theorem implies that the transformations of the inversive plane $\widehat{\mathbb{C}}$ are composites of inversions in inversive lines; in fact, every inversive transformation can be factored into four or fewer inversions. A transformation T that factors into two or four inversions is called *direct* (orientation-preserving), whereas a transformation T that factors into one or three inversions is called *opposite* (orientation-reversing).

2) Every direct inversive transformation can be represented as a linear fractional transformation (LFT):

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Here $T(\infty) = \frac{a}{c}$ and $T\left(-\frac{d}{c}\right) = \infty$. We typically represent a LFT by the complex projective matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is consistent with the fact that the inverse of T is represented by the inverse of the matrix and that the composition $T_2 \circ T_1$ is represented by the matrix product $T_2 T_1$.

3) The image of an inversive line under a LFT is obtained by spin conjugation: Let M be a complex projective matrix and let M^* be the complex conjugate of its transpose; if H is Hermitian then MHM^* is called its *spin conjugate* by M. Note that MHM^* is also Hermitian. Spin conjugation provides the following theorem:

> If $\det H < 0$ then the image of the inversive line represented by H under the LFT T is $T_{co} H T_{co}^*$, where T_{co} is the cofactor matrix of T.

4) The theorem of Riemann and Möbius implies that the inversive group $\mathcal{M}$ consists of direct transformations (each of which is a LFT) and opposite transformations (each of which is the composition of a LFT with complex conjugation). In particular, the transformation $z \mapsto \overline{z}$ is the composition of the identity LFT ($a = d, b = c = 0$) with complex conjugation (reflection in the real line). Note that reflection of an inversive line H in the real line produces the inversive line H^t, the transpose of H. This fact, together with **3)**, allows us to easily transform any inversive line by a member of $\mathcal{M}$.

V. Important skills.

1. Represent an inversive line by a Hermitian matrix and be able to determine the Cartesian form of an inversive line from its matrix.

2. Compute the inner product of two Hermitian matrices and, if they represent inversive lines determine whether they intersect and at what angle.

3. Find the image of points and inversive lines under inversion in a given inversive line.

4. Use spin conjugation to find the image of an inversive line under an inversive transformation.

5. Determine whether a given inversive line is a member of the pencil generated by two other inversive lines.

First Project Assignment

Do one of the following. Presentation must be in a typed format and may be submitted electronically.

1. Let H_1 and H_2 be Hermitian matrices and let $H_1' = TH_1T^*, H_2' = TH_2T^*$, where T is the matrix of a LFT. Show that $\frac{\langle H_1, H_2\rangle}{|H_1||H_2|} = \frac{\langle H_1', H_2'\rangle}{|H_1'||H_2'|}$. Suppose H_1 and H_2 represent intersecting inversive lines and let $\cos\alpha = \frac{\langle H_1, H_2\rangle}{|H_1||H_2|}$. Show that α is the angle at which these inversive line intersect. Conclude that LFTs act conformally on inversive lines.

2. Let $ax + by + cz = 0$ be a 2-space Π in $\mathbb{R}^3$. Determine conditions on a, b, c so that Π is a secant to the cone $x^2 + y^2 - z^2 = 0$. In this case let L be the intersection of Π with the upper branch of the hyperboloid $x^2 + y^2 - z^2 = -1$, and let H be the stereographic projection from $(0, 0, -1)$ of L into the unit disk. Find the Hermitian matrix for H.

Projective and Inversive Models

A. The *real projective plane* is the geometry whose *points* are the 1-spaces of $\mathbb{R}^3$ and whose *lines* are the 2-spaces of $\mathbb{R}^3$. It follows that any two points determine a unique line and any two lines intersect in a unique point. The affine Euclidean plane is obtained as a sub-geometry by removing any line and all of its points, so it is now possible for two lines to be parallel. An easy way to see this is to select an affine plane Π in $\mathbb{R}^3$, i.e., a plane that does not contain the origin. Every point of Π is its intersection with some 1-space in $\mathbb{R}^3$, and every 1-space intersects Π unless it is contained in the unique 2-space parallel to Π. This 2-space is the so-called line at infinity.

B. The *inversive plane* is the geometry whose *points* are in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and whose *lines* are all possible circles and straight lines in the complex plane. This model is obtained by stereographic projection of the unit sphere S^2 into the plane from a pole, one of its points such as $(0, 0, 1)$. Thus the inversive plane can also be viewed as the Riemann sphere with lines determined by planes that intersect the sphere in more that one point, hence circles. Circles that contain the pole project to straight lines in $\widehat{\mathbb{C}}$, that is, inversive lines that pass through ∞. It is convenient to represent the lines of the inversive plane by 2×2 Hermitian matrices H with negative determinant, whereby the points on the line are the solutions of $ZHZ^* = 0$ where $Z = \begin{bmatrix} z & 1 \end{bmatrix}$. In this representation the Hermitian matrices with zero determinant correspond to points of the inversive plane. Note, however, representation of points and lines is not unique because H and λH represent the same object if λ is a non-zero real number, that is, points and lines are 1-spaces in the four-dimensional real vector space of Hermitian matrices. The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has no representational value since it does not span a 1-space, much as the homogeneous triple $[0, 0, 0]$ does not span a 1-space in $\mathbb{R}^3$ and so does not represent a point in the projective plane.

The Möbius Group $\mathcal{M}$

The group $\mathcal{M}$ of transformations of the inversive plane is generated by inversions in lines. Inversion in a line through ∞ coincides with Euclidean reflection in the straight line. For a circle of radius ρ centered at O the inversion of P is the unique point P' on the ray $\overrightarrow{OP}$ such that $(OP)\,(OP') = \rho^2$. If O and P are represented by the complex numbers z_0 and z, respectively, then it is straightforward to show that P' is given by

$$z_0 + \frac{\rho^2}{\overline{z} - \overline{z}_0} = \frac{z_0\overline{z} + \left(\rho^2 - |z_0|^2\right)}{\overline{z} - \overline{z}_0}.$$

Note that this is the transformation $T \circ \tau$, where $\tau(z) = \overline{z}$ and $T = \begin{bmatrix} z_0 & \rho^2 - |z_0|^2 \\ 1 & -\overline{z}_0 \end{bmatrix}$.

Since the Hermitian matrix for the circle of inversion is $H = \begin{pmatrix} 1 & -\overline{z}_0 \\ -z_0 & |z_0|^2 - \rho^2 \end{pmatrix}$

it follows that $T = JH$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This formula also applies if H represents a line through ∞.

The product of inversion in H_1 followed by inversion in H_2 should therefore result in a LFT. In fact, since $T_1 = JH_1$ and $T_2 = JH_2$ it is easy to check that $T_2 \circ \tau \circ T_1 \circ \tau$ is the LFT represented by the matrix

$$\left(H_1 H_2^{adj}\right)^t .$$

For example, inversion in the unit circle $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ followed by inversion in $\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$, the circle of radius 2 centered at 0, is the LFT $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, i.e., $z \mapsto 4z$, whereas reversing the order of inversions results in $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, i.e., $z \mapsto \frac{1}{4}z$. In general, the product of an even number of inversions will result in a LFT, whereas the product of an odd number of inversions is complex conjugation followed by a LFT. The LFTs comprise a normal subgroup of index 2 in $\mathcal{M}$, sometimes called the *direct* Möbius group. The other coset consists of the opposite (orientation reversing) transformations.

Transformation of the Inversive Plane by $\mathcal{M}$

Let z be on the inversive line H and let $T(z) = w$. Since $ZHZ^* = 0$ and $Z = WT_{co}$, where T_{co} is the cofactor matrix of T, it follows that $(WT_{co})\, H\, (WT_{co})^* = W\left(T_{co}HT_{co}^*\right)W^* = 0$. Note that $H' = T_{co}HT_{co}^*$ is Hermitian and has negative determinant. Thus H' is the matrix representing the image of H under T. This shows in particular that LFTs take inversive lines to inversive lines. On the other hand, it is easy to show that the complex conjugate of an inversive line H is the inversive line $\overline{H} = H^t$, so the group $\mathcal{M}$ consists of collineations of the inversive plane. Möbius and Riemann showed that $\mathcal{M}$ is the full group of inversive collineations.

As a corollary to this transformation rule we can find the Hermitian representation of the point ∞ in $\widehat{\mathbb{C}}$. Recall that $\begin{pmatrix} 1 & -\overline{z}_0 \\ -z_0 & |z_0|^2 \end{pmatrix}$ is the point z_0 in $\mathbb{C}$, so $z_0 = 0$ is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Evidently ∞ is the image of 0 under the LFT $T(z) = \frac{1}{z}$. Since $T_{co} = T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the point ∞ must be $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Note that this is consistent with the image of z_0 under inversion in any circle centered at z_0.

Properties of Inversion

Inversion and orthogonality are closely related. For example, here is an easy exercise in the inversive plane that proves an interesting result from Euclidean geometry:

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles that intersect at A and B. If there is a line through A but not through B whose chords with $\mathcal{C}_1$ and $\mathcal{C}_2$ each have length AB then $\mathcal{C}_1$ and $\mathcal{C}_2$ are orthogonal. (**Suggestion:** Invert the line and the two circles in the circle of radius AB centered at A.)

It is clear that reflection in a straight line is anti-conformal. Using Steiner's construction it is not difficult to show that inversion in a circle of finite radius is also anti-conformal and that it transforms inversive lines into inversive lines. This shows that the full Mobius group $\mathcal{M}$ consists of conformal and anti-conformal collineations of the inversive plane. Inversion is also closely related to stereographic projection. Here are two straightforward exercises:

1) *Let z and z' be inversions of each other in the unit circle, and let P and P' be their respective stereographic projections from $(0,0,1)$ onto S^2. Then P' is the reflection of P in the plane of the unit circle.*

2) *Let $P=(u,v,w)$ be on the unit sphere and let P^o be its antipodal point, i.e., $P^o=(-u,-v,-w)$. If z and z^o are the stereographic projections of P and P^o, respectively, into the plane then the product zz^o is on the unit circle.*

Note that if two inversive lines are orthogonal then each is invariant upon inversion in the other. The following theorem is sometimes referred to as the fundamental *characterization of inversion:*

Let H be an inversive line. Distinct points P and Q are inversive images in H iff the every member of their elliptic pencil is orthogonal to H.

Proof. Let P and Q be inversive images in H.

i) If K is an inversive line that contains P and Q then there exists $A \in K \cap H$, and $PAQ \longleftrightarrow QAP$, so K is invariant under inversion in H. But then $K \perp H$ since angles on either side of A are equal in magnitude.

ii) Conversely, let $K_1 \perp H$ and $K_2 \perp H$ be members of the elliptic pencil through the distinct points P, Q.

Then $K_1 \cap K_2 = \{P, Q\} = K_1' \cap K_2'$, where K_1' and K_2' are the respective inversions of K_1 and K_2 in H. But neither P nor Q is on H since they are distinct, so it must be that P and Q are inversive images in H. ■

G_D, the group of collineations of $\mathcal{D}$

Find product of two inversions: Let L be a line of $\mathcal{D}$ and identify L with its Hermitian matrix. Reflection in L interchanges the two half-planes of $\mathcal{D}$, and the product of reflection in L_1 followed by reflection in L_2 should result in a LFT. In fact, with $T_1 = JL_1$ and $T_2 = JL_2$ it is easy to check that $T_2 \circ \tau \circ T_1 \circ \tau$ is the LFT represented by the matrix

$$\left(L_1 L_2^{adj}\right)^t .$$

In particular, reflection in $L_1 = \begin{pmatrix} 1 & \alpha \\ \alpha^* & 1 \end{pmatrix}$ followed by reflection in $L_2 = \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix}$ is $T = \begin{bmatrix} \alpha\beta^* - 1 & \beta^* - \alpha^* \\ \beta - \alpha & \alpha^*\beta - 1 \end{bmatrix}$. Let $a = \alpha\beta^* - 1, b = \beta^* - \alpha^*$. Then it is easy to show that $|a| > |b|$.

Exercise:

1) Find the product of two inversions if one or both of L_j passes through 0.

2) If $|a| > |b|$, show that $T = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$ is the product of two reflections in lines of $\mathcal{D}$.

3) Show that the group of LFTs preserving $\mathcal{D}$ is $\left\{ T = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix} : |a| > |b| \right\}$.

4) Find a, b, with $|a| > |b|$, such that $T \circ \tau$ is not a single reflection. Conclude that any member of G_D is the product of 3 or fewer reflections.

The Subgroup of Direct Collineations

Every orientation-preserving (direct) member of G_D is either a **rotation, limit rotation, or translation** depending on whether it is the product of reflections in **intersecting, parallel, or ultra-parallel** lines, respectively. Let $T = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$ be a direct transformation, so $|a| > |b|$. The factorization of T into reflection in L_1 followed by reflection in L_2 is not unique, but the incidence relation between L_1 and L_2 is invariant, and determined by the location of the fixed points of T. For example, if $b = 0$ then T is a rotation about 0, and its other fixed point is ∞, the inversion of 0 in U. We can let $\frac{a}{a^*} = e^{i\theta}$, whereby T is the product of reflections in L_1, L_2 through 0 and the angle from L_1 to L_2 is $\frac{1}{2}\theta$.

If $b \neq 0$, let $L_1 = \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix}$, so that the product of the reflections is

$$\left[L_1 L_2^{adj}\right]^t = \begin{bmatrix} -\beta\alpha^* & -\alpha^* \\ -\alpha & -\alpha\beta^* \end{bmatrix}$$

which is T provided

$$\begin{aligned} \alpha &= -b^* \\ \beta &= \frac{a}{b} \end{aligned}$$

The fixed points of T are found by solving $T(z) = z$, which yields the quadratic equation

$$b^* z^2 + (a^* - a)\, z - b = 0$$

and thus the solutions

$$z = \frac{1}{2b^*}\left(a - a^* \pm \sqrt{(a - a^*)^2 + 4\,|b|^2}\right)$$

which are paired by inversion in U. Thus, if one of the fixed points is in $\mathcal{D}$ then the other is not, and T is a rotation about the point in $\mathcal{D}$.

Exercises:

1) Let $\Delta = (a - a^*)^2 + 4\,|b|^2$.

a) If $\Delta = 0$ then T has a single fixed point of multiplicity 2. Show this fixed point is on U, and so T is a limit rotation.

b) If $\Delta < 0$ show that T is a rotation.

c) The third possibility is two fixed points on the unit circle. Show that $\Delta > 0$ in this case, which is a translation.

2) We can also factor $T = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ \frac{b^*}{a^*} & 1 \end{bmatrix}$. Show that the factor on the right is a translation. This is the **Canonical Form Theorem:**

A direct transformation is the product of a translation followed by rotation about 0.

3) Provide an a geometric interpretation in $\widehat{\mathbb{C}}$ of the alternative factorization $T = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{a^*}{b^*} \\ \frac{a}{b} & 1 \end{bmatrix}$. Note that neither factor is a transformation in G_D.

$\mathcal{M}$-Invariance of Inversion

Let H be an inversive line and let $H(z)$ denote the inversion of z in H. If K is an inversive line let $K(H)$ denote the inversion of H in K. *The inversions of z and $H(z)$ in K are then inversions of each other in $K(H)$.* This fact implies that inversion is preserved by the action of the Mobius group $\mathcal{M}$. Its proof is usually based on the (anti)conformal action of inversion, but we can show it algebraically as a consequence of spin conjugation on the space of Hermitian matrices.

Using the above notation, the statement can be written

$$K\left(H\left(z\right)\right)=K(H)\left(K(z)\right)$$

which will follow from two simple observations. First, if $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and M is any 2×2 matrix note that

$$JMJ^{-1}=M_{co}$$

This explains the importance of the matrix J: *Conjugation by J produces the cofactor matrix of M.* Further, if H is an inversive line then $H_{co}=JHJ$ because $J^{-1}=-J$ and multiplying H by a non-zero real number results in the same inversive line. Second, note that

$$\begin{aligned}(JH)_{co} &= HJ\\ (JH)^{*}_{co} &= -JH\end{aligned}$$

because H is Hermitian. It follows that $K(H)=\left(T_K\right)_{co}H^t\left(T_K\right)^{*}_{co}=$

$$\begin{aligned}&\left(KJ\right)H^t\left(-JK\right)\\ =\ &K\left(JH^tJ\right)K\\ =\ &KH^t_{co}K\\ =\ &KH^{adj}K\end{aligned}$$

which shows that spin conjugation takes a particularly simple form when it is representing inversion of one line in another.

Now let Z be any real-projective Hermitian matrix; for example, in the statement we are trying to prove we would have $Z=\begin{pmatrix}1 & -z^*\\ -z & zz^*\end{pmatrix}$ so that it represents the point z. Then

$$\begin{aligned}&K\left(H\left(Z\right)\right)\\ =\ &KJ\left(HJZ^tJH\right)^tJK\\ =\ &K\left(JH^tJ\right)Z\left(JH^tJ\right)K\\ =\ &KH^{adj}ZH^{adj}K\end{aligned}$$

On the other hand

$$\begin{aligned} K(H)\,(K(Z)) &= \left(KJH^tJK\right)J\left(KJZ^tJK\right)^tJ\left(KJH^tJK\right) \\ &= KH^{adj}\left(KK^{adj}\right)Z\left(K^{adj}K\right)H^{adj}K \\ &= KH^{adj}ZH^{adj}K \end{aligned}$$

where the last step follows because $KK^{adj} = K^{adj}K$ is a scalar matrix. Thus $K\left(H\left(Z\right)\right) = K(H)\,(K(Z))$. ■

The upper half-plane model $\mathcal{U}$

Let $\mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and let $T : \mathcal{D} \to \mathcal{U}$, for example, $T = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$. Let L be a line of $\mathcal{D}$.

If $L = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$, $0 \leq \theta < \pi$, then $T(L) =$

$$\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

which is the portion of the Cartesian curve

$$\begin{cases} (x-\cot\theta)^2 + y^2 = \csc^2\theta \\ \quad x = 0, \text{ if } \theta = 0 \end{cases}$$

in $\mathcal{U}$.

If $L = \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix}$, $\beta = B + iC : B^2 + C^2 > 1$, then $T(L) =$

$$\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{pmatrix} i\beta - i\beta^* + 2 & \beta + \beta^* \\ \beta + \beta^* & -i\beta + i\beta^* + 2 \end{pmatrix}$$

which is the portion of the Cartesian curve

$$\begin{cases} \left(x - \frac{B}{C-1}\right)^2 + y^2 = \frac{B^2+C^2-1}{(C-1)^2}, \text{ if } C \neq 1 \\ \quad x = -\frac{1}{B}, \text{ if } C = 1 \end{cases}$$

in $\mathcal{U}$. Together, these curves represent all inversive lines orthogonal to the real line in $\widehat{\mathbb{C}}$.

G_U, the group of collineations of $\mathcal{U}$

The group G_U is the conjugate in $\mathcal{M}$ by T of the group G_D. Thus, if $M = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix}$ is a LFT in G_D, then $TMT^{-1} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} =$ $\begin{bmatrix} \alpha - i\beta + \alpha^* + i\beta^* & -i\alpha + \beta + i\alpha^* + \beta^* \\ i\alpha + \beta - i\alpha^* + \beta^* & \alpha + i\beta + \alpha^* - i\beta^* \end{bmatrix} = \begin{bmatrix} \operatorname{Im}(\beta) + \operatorname{Re}(\alpha) & \operatorname{Im}(\alpha) + \operatorname{Re}(\beta) \\ -\operatorname{Im}(\alpha) + \operatorname{Re}(\beta) & -\operatorname{Im}(\beta) + \operatorname{Re}(\alpha) \end{bmatrix}$ is a LFT in G_U. Thus $TMT^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}, ad - bc > 0$. It follows that $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}$ is the subgroup of direct collineations in G_U because, given such a, b, c, d, it is possible to find α and β such that $\operatorname{Im}(\beta) + \operatorname{Re}(\alpha) = a, \operatorname{Im}(\alpha) + \operatorname{Re}(\beta) = b, -\operatorname{Im}(\alpha) + \operatorname{Re}(\beta), -\operatorname{Im}(\beta) + \operatorname{Re}(\alpha) = d$. (Show this.)

What are the opposite collineations in G_U ? These will be the compositions of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\tilde{\tau}$, the conjugate of τ by T. However, τ is reflection in the line $M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ of $\mathcal{D}$, so $\tilde{\tau}$ is reflection in the line $T(M)$ of $\mathcal{U}$, and $T(M)$ is the unit circle $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus $\tilde{\tau}(z) = \frac{1}{z^*}$, from which it follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ \tilde{\tau}(z) = \frac{bz^* + a}{dz^* + c} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \circ \tau(z)$. The factorization on the right decomposes the collineation into two transformations that each interchange the upper and lower half-planes. Note that we can also express an opposite collineation in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ \tau'$, where $\tau'(z) = -z^*$, the reflection of z in the imaginary line. This last decomposition is used frequently.

Exercise: Let L be any line of $\mathcal{U}$. If H represents either a point or line of $\mathcal{U}$, find its reflection H' in L. (Recall that $H' = LH^{adj}L$.)

Distance in $\mathcal{D}$

If points A, B are represented by the complex numbers a, b in the disk, the distance $d(A, B)$ between them should be measured by a real-valued function $s(a, b)$ with the following properties:

1) $s(a, b) \geq 0$ and $s(a, b) = 0$ iff $a = b$

2) $s(a, b) = s(b, a)$

3) $s(a, b) \leq s(a, c) + s(c, b)$ and $s(a, b) = s(a, c) + s(c, b)$ iff c is on the segment between a and b

Further, we have seen that G_D is generated by reflections, which should be isometries of $\mathcal{D}$, and so we should have the additional property:

4) $s(T(a), T(b)) = s(a, b)$ for all $T \in G_D$

To find such a function s, let $s(z) := s(0, z) = s(z, 0)$ and note that, from property **4**, $s(-z) = s(z)$. It suffices to determine $s(z)$ and then use property **4** again to deduce the general case. In fact, it suffices to assume $z \in \mathbb{R}$ with $0 \leq z < 1$. Assuming the function $z \mapsto s(z)$ is differentiable on $(0, 1)$ and without critical points, let $h \in \mathbb{R}$ with $0 \leq h < 1$, and let T be the translation

$$T(z) = \frac{z + h}{hz + 1}$$

Then $T(0) = h$ and $T(-h) = 0$. Since 0 is between $-h$ and z, property **3** yields

$$s(-h, z) = s(-h, 0) + s(0, z) := s(-h) + s(z) = s(h) + s(z).$$

However, $s(T(z)) := s(0, T(z)) = s(T(-h), T(z)) = s(-h, z) = s(h) + s(z)$. It follows that

$$s(h) + s(z) = s\left(\frac{z + h}{hz + 1}\right).$$

Since s is locally invertible with differentiable inverse (no critical points), we can form the difference quotient for the function $g = s^{-1}$ at $s(z)$

$$\begin{aligned} (*) \quad : \quad & \frac{g\left(s(z) + s(h)\right) - g(s(z))}{s(h)} \\ = \quad & \frac{\frac{z+h}{hz+1} - z}{s(h)} = \frac{h}{s(h)} \left(\frac{1 - z^2}{1 + hz}\right) \end{aligned}$$

and take the limit as $s(h) \to 0$ (which implies $h \to 0$). The limit of the left side of $(*)$ is, by definition, $g'(s(z))$ and the limit of the right side is $\left(1 - z^2\right) \lim\limits_{h \to 0} \frac{1}{\frac{s(h) - s(0)}{h}}$, which is

$$\frac{1 - z^2}{s'(0)}$$

It follows that

$$(**) : \left(s^{-1}\right)' (s(z)) = \frac{1 - z^2}{s'(0)}$$

which can be solved for $s'(z)$ using the **Inverse Function Theorem:** *If f is differentiable* with $f(a) = b$ *and* $f'(a) \neq 0$ *then*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}.$$

.

To apply the IFT, let $f = s$ and $a = z$ so that $b = s(z)$. Then, from $(**)$

$$\begin{aligned} \frac{1-z^2}{s'(0)} &= \frac{1}{s'(z)} \\ s'(z) &= \frac{s'(0)}{1-z^2}. \end{aligned}$$

This differential equation has a unique solution if the values $s(0)$ and $s'(0)$ are specified. Now $s(0) = 0$, but $s'(0)$ is not specified. A consistent definition of distance can be obtained from any positive value of $s'(0)$ (since s is increasing at 0), whereby it is convenient to set $s'(0) = 1$. Then

$$s(z) = \tanh^{-1} z.$$

This determines $s(0, z)$ for any non-negative real z. To find $s(0, z)$ for any z in the disk apply property **4** to any rotation about 0. Then $s(0, z) = \tanh^{-1}|z|$. Finally, if w is any point in the disk, let

$$T(z) = \frac{z-w}{1-w^*z}$$

Then $s(z, w) = s(T(z), T(w)) = s\left(\frac{z-w}{1-w^*z}, 0\right)$, and so

$$s(z, w) = \tanh^{-1}\left|\frac{z-w}{1-w^*z}\right| = \tanh^{-1}\left|\frac{w-z}{1-z^*w}\right| = s(w, z).$$

The triangle inequality in property **3** can now be verified directly. WLOG we can assume that $a \in (0, 1)$, $c = 0$, and $b = r\cos\theta$, $0 \leq r < 1$, and then show $s(a, 0) + s(0, b) \geq s(a, b)$, *i.e.*,

$$\tanh^{-1} a + \tanh^{-1} r \geq \tanh^{-1}\left|\frac{b-a}{1-ab}\right|$$

Apply tanh to both sides of this inequality and let $p = \frac{a+r}{1+ar}, q = \left|\frac{b-a}{1-ab}\right|$. To show that $p \geq q$, note that

$$\begin{aligned} p^2 - q^2 &= \frac{(a+r)^2 |1-ab|^2 - |b-a|^2 (1+ar)^2}{|1-ab|^2 (1+ar)^2} \\ & 2ar(1+\cos\theta)\frac{\left(1-a^2\right)\left(1-r^2\right)}{|1-ab|^2(1+ar)^2} \end{aligned}$$

Thus $p^2 - q^2 \geq 0$. Further, $s(a,b) = s(a,0) + s(0,b)$ iff $r = 0$ or $\theta = \pi$, which is in agreement with **3**.

Exercise:

For any $z \in \mathbb{C}$, let

$$B(z) = \frac{2z}{1 + |z|^2}$$

Show that B is a two-to-one map that sends z and its inversion in U to the same point. If $z \in \mathcal{D}$, show that $s(B(z),0) = 2s(z,0)$. If L is a line in $\mathcal{D}$, show that $B(L)$ is the inversive line through ∞ and the boundary points of L.

Circles in $\mathcal{D}$

Let $\mathcal{C}(w,k)$ be a circle of hyperbolic radius k centered at w. Then $T = \begin{bmatrix} 1 & -w \\ -w^* & 1 \end{bmatrix}$ maps $\mathcal{C}(w,k)$ to $\mathcal{C}(0,k)$. Let $\tanh k = q$. Then $\mathcal{C}(0,k) = \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix}$ and so $\mathcal{C}(w,k) =$

$$\begin{aligned} & \begin{bmatrix} 1 & -w^* \\ -w & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix} \begin{bmatrix} 1 & -w^* \\ -w & 1 \end{bmatrix} \\ = & \begin{pmatrix} 1 - q^2 |w|^2 & \left(q^2 - 1\right) w^* \\ \left(q^2 - 1\right) w & |w|^2 - q^2 \end{pmatrix} \end{aligned}$$

since $T^{-1} = \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}$. It follows that the inversive center γ and the inversive radius ρ are

$$\begin{aligned} \gamma &= \frac{\left(1 - q^2\right)}{1 - q^2 |w|^2} w \\ \rho &= q \frac{1 - |w|^2}{1 - q^2 |w|^2} \end{aligned}$$

Conversely, if an inversive line is contained within the open unit disk then it will be represented by the Hermitian matrix

$$\begin{pmatrix} 1 & -\gamma^* \\ -\gamma & |\gamma|^2 - \rho^2 \end{pmatrix}$$

where $|\gamma| + \rho < 1$ since this is a circle with inversive center γ and inversive radius ρ. To find its hyperbolic center w and hyperbolic radius k we first note that

$$k = \frac{1}{2} s \left(\gamma + \rho \frac{\gamma}{|\gamma|}, \gamma - \rho \frac{\gamma}{|\gamma|} \right)$$

and then use rotation about 0 to note that

$$\begin{aligned} k &= \frac{1}{2}s\left(|\gamma|+\rho,|\gamma|-\rho\right) \\ &= \frac{1}{2}\tanh^{-1}\left|\frac{2\rho}{1-|\gamma|^2+\rho^2}\right| \end{aligned}$$

Thus $\tanh 2k = \frac{2q}{1+q^2} = \left|\frac{2\rho}{1-|\gamma|^2+\rho^2}\right| = \frac{2\rho}{1-|\gamma|^2+\rho^2}$, since $1-|\gamma|^2+\rho^2 > 0$. To find w we use the fact, from above, that $\rho = q\frac{1-|w|^2}{1-q^2|w|^2}$ along with the fact that $w = c\gamma$ with $c > 0$. Consequently, $\rho = q\frac{1-c^2|\gamma|^2}{1-q^2c^2|\gamma|^2}$ from which we obtain $c = \frac{1}{|\gamma|}\sqrt{\frac{q-\rho}{q(1-q\rho)}}$. As expected, $c > 1$ (since $q < 1$). We conclude that the inversive line is the hyperbolic circle $\mathcal{C}(w,k)$ where

$$\begin{aligned} k &= \frac{1}{2}\tanh^{-1}\left(\frac{2\rho}{1-|\gamma|^2+\rho^2}\right) \\ w &= \sqrt{\frac{q-\rho}{q\left(1-q\rho\right)}}\frac{\gamma}{|\gamma|} \end{aligned}$$

Another interesting identity that relates the inversive and hyperbolic data for the circle to each other is obtained from the ratio

$$\frac{\gamma}{\rho} = \frac{1}{q}\frac{1-q^2}{1-|w|^2}w$$

Since $q = \tanh k$ it follows that $\frac{2q}{1-q^2} = \sinh 2k$, and thus, if $|w| = \tanh h$ then

$$\rho \sinh 2h = |\gamma| \sinh 2k$$

Inversive lines as orbits
Angle-of-parallelism

When a group acts on a set the collection of images obtained from a given point is called the *orbit* of the point by the group. Orbits of points in $\mathcal{D}$ by certain subgroups of G_D are inversive lines, as the following important examples show.

1. Circle: rotation orbit

Let T be a rotation about the point w and let $s(z,w) = k$. Then $\mathcal{C}(w,k)$ is the orbit of z by the group of all such T. Note that this group is the stabilizer of w in the group of direct collineations. The orbit of any line through w is a concurrent pencil in the hyperbolic plane, afforded by the elliptic inversive pencil orthogonal to the hyperbolic pencil whose point circles are w and $\frac{1}{w^*}$.

2. Horocycle: limit rotation orbit

A *horocycle* is an inversive line in the unit disk that meets U at a single point. For example, if $H = \begin{pmatrix} 1 & -r \\ -r & 2r-1 \end{pmatrix}$, $0 < r < 1$, then H is a horocycle at $z = 1$, and any horocycle at 1 takes this form for some r. The LFTs $T_t = \begin{bmatrix} 1+it & -it \\ it & 1-it \end{bmatrix}$, $t \in \mathbb{R}$, form a group, the group of limit rotations that fix $z = 1$. Note that $T_t(H) = H$, and so, if $z \in H$, then H is the orbit of z by this group. The orbit of any line through $z = 1$ is a parallel pencil in the hyperbolic plane, afforded by the parabolic inversive pencil orthogonal to the pencil containing the horocycles. Note that if $H_r = \begin{pmatrix} 1 & -r \\ -r & 2r-1 \end{pmatrix}$ and $H_s = \begin{pmatrix} 1 & -s \\ -s & 2s-1 \end{pmatrix}$, with $0 < s < r < 1$, then $\frac{r-s}{r+s-2rs} = \tanh d$, where d is the distance between these two horocycles as measured along the real line (which is orthogonal to any horocycle at 1). Measuring the distance along any other line of the parallel pencil must produce the same value d. *We conclude that the distance between two horocycles with the same boundary point is constant.*

3. Equidistant curve: translation orbit

The intersection of $\mathcal{D}$ with an inversive line that meets U at two distinct points is called an *equidistant curve.* Thus, equidistant curves include the lines of $\mathcal{D}$. For example, if $|b| < 1, b \in \mathbb{R}$, then $T = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ is a translation that leaves the real line invariant. Note that T fixes ±1 so all such T form a group. Other than the real line, any equidistant curve meeting U at ±1 is of the form $H = \begin{pmatrix} 1 & iC \\ -iC & -1 \end{pmatrix}$, $C \neq 0$. It follows that $T(H) = H$ and so, if $z \in H$, then H is the orbit of z by this group of translations. The orbit of any line perpendicular to the real line is an ultra-parallel pencil, afforded by the hyperbolic inversive pencil orthogonal to the elliptic pencil through $z = 1$ and $z = -1$.

Suppose $z = i \tanh u$ with $u \in \mathbb{R}$, so that the distance between z and the real line is $s(z, 0) = |u|$. Then the perpendicular through $T(z)$ must intersect the real line at a point w such that $s(T(z), w) = |u|$. Thus the additivity property of distance along a line shows that *the distance between any two equidistant curves with the same boundary points is constant.* It follows that the locus of points that are a given distance from a line is a pair of equidistant curves that share boundary points with the line. This is the hyperbolic version of the Euclidean theorem that the locus of such points is two lines parallel to the given line, one in each half-plane.

It is useful to characterize equidistant curves by the angle α of intersection with the given line L, where $0 \leq \alpha < \frac{\pi}{2}$. Without loss of generality we can let $L = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. If the equidistant curve $H = \begin{pmatrix} 1 & iC \\ -iC & -1 \end{pmatrix}$ intersects the imaginary line at $z = i \tanh u$ then $C = -\operatorname{csch} 2u$. It follows that

$$\cos \alpha = \operatorname{sech} 2u$$

from which we also obtain

$$\begin{aligned} \sin \alpha &= \tanh 2u \\ \tan \alpha &= \sinh 2u \end{aligned}$$

Now let ϕ be the angle between the imaginary line and either line through z that is parallel to L. Since the equidistant curve through z makes the same angle with this parallel that it makes with L it follows that $\alpha + \phi = \frac{\pi}{2}$, that is, they are complementary angles. We conclude

$$\tan \phi = \cot \alpha = \operatorname{csch} 2u$$

or, equivalently,

$$\begin{aligned} \sin \phi &= \operatorname{sech} 2u \\ \cos \phi &= \tanh 2u \end{aligned}$$

The angle ϕ is called the *angle of parallelism* determined by the hyperbolic distance $|u|$. If $\phi = \frac{\pi}{4} = \alpha$ the equidistant curve is called a *right curve*; in this case any point on the curve forms a doubly asymptotic right triangle withe the boundary points of L.

Second Project Assignment
Due: June 13
Do one of the following seven. Presentation must be in a typed format and may be submitted electronically.

1. Canonical form of direct collineations.
Let $T = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 1 & m \\ m^* & 1 \end{bmatrix}$, $|m| < 1$. Show that T is a rotation if $|m| < |\sin\theta|$, a limit rotation if $|m| = |\sin\theta|$, and a translation if $|m| > |\sin\theta|$. Use this fact to show that the translations are a subgroup of G_D.

2. Manning's construction.
Let P be a point not on the line L and let Q be the foot of the perpendicular to L through P. Let S be a point on the line through P perpendicular to line PQ such that the length of segment PS is the length of segment PQ. Let R be the foot of the perpendicular to L through S and let T be the point on segment SR such that the length of segment PT is the length of segment QR. Show that line PT is parallel to L.

3. Cross-ratio and hyperbolic distance.
In $\widehat{\mathbb{C}}$, the *cross-ratio* of the ordered quadruple $(z_1, z_2; z_3, z_4)$ is defined to be

$$\frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

Show that the cross-ratio is taken to its complex conjugate under inversion, thus it is invariant under direct transformations. Then use this fact to show that the hyperbolic distance between z_1 and z_2 in $\mathcal{D}$ is

$$d(z_1, z_2) = \frac{1}{2} \ln |(z_1, z_2; \zeta_1, \zeta_2)|,$$

where ζ_1 and ζ_2 are the boundary points of the line through z_1 and z_2 arranged in the order $\zeta_2, z_1, z_2, \zeta_1$. **Suggestion:** Transform the line to the real axis and send one of the z_j to 0.

4. Distance in $\mathcal{U}$

Let T be a LFT that maps $\mathcal{U}$ to $\mathcal{D}$. Show that any such T is of the form $T_u = \begin{bmatrix} 1 & u^* \\ 1 & u \end{bmatrix}$, where u is a point in $\mathcal{U}$, followed by a rotation about 0. Given z and w in $\mathcal{U}$, let $Z = T(z)$ and $W = T(w)$. Show that $\left|\frac{Z-W}{1-Z^*W}\right|$ is independent of the choice of u, and therefore independent of the choice of T. Use this conclusion to find a formula for the distance between z and w in $\mathcal{U}$.

5. Horocycles and angle-of-parallelism.

Let L be a line and let E be an equidistant curve for L with common boundary point A. Let L' be a line perpendicular to L and let $P = E \cap L'$. If H is the horocyle at A that contains P show that the angle between H and L' is the complement of the angle of parallelism at P with respect to L. **Suggestion:** First show that it suffices to take L to be the imaginary line and L' the real line. Then find a LFT that interchanges H with E and L' with L.

6. Split inversion.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the open half-planes determined by the line L in $\mathcal{D}$, and for $P \in \mathcal{P}_1 \cup \mathcal{P}_2$ let L_P be the perpendicular to L through P. If $P \in \mathcal{P}_j$ the *split inversion of* P *in* L is the point $P' \in L_P \cap \mathcal{P}_j$ such that the angle of parallelism to L at P' is the complement of the angle of parallelism to L at P. Show that P' is the inversion of P in the right curve for L contained in P_j, so split inversion transforms $\mathcal{P}_1$ and $\mathcal{P}_2$ onto themselves.

7. A special isosceles triangle.

Show there is a unique (up to congruence) isosceles triangle whose altitudes are parallel and whose $\perp$-bisectors are also parallel. Find its angles and side lengths.

Triangles in $\mathcal{D}$

Let POQ be the triangle with $O = 0, P = p, Q = qe^{i\theta}$. Assume $0 < p, q < 1$ and $0 < \theta < \pi$. Thus the length of side PQ is

$$\begin{aligned} c &= \tanh^{-1}\left|\frac{p - qe^{i\theta}}{1 - pqe^{i\theta}}\right| \\ &= \tanh^{-1}\sqrt{\frac{p^2 - 2pq\cos\theta + q^2}{p^2q^2 - 2pq\cos\theta + 1}} \end{aligned}$$

Note that if $s = \tanh^{-1} r$ then $\cosh 2s = \frac{1+r^2}{1-r^2}$ and $\sinh 2s = \frac{2r}{1-r^2}$. Let $a = \tanh^{-1} p$ and $b = \tanh^{-1} q$. It follows that

$$\cosh 2c = \cosh 2a \cosh 2b - \sinh 2a \sinh 2b \cos\theta.$$

This is the hyperbolic *Law of Cosines*. If $\theta = \frac{\pi}{2}$ we obtain the hyperbolic *Pythagorean Theorem*

$$\cosh 2c = \cosh 2a \cosh 2b.$$

This law approaches the Euclidean law in the limit as $a \to 0, b \to 0$. This is because

$$\begin{aligned} \cosh 2s &= 1 + 2s^2 + \cdots \\ \sinh 2s &= 2s + \cdots \end{aligned}$$

and so

$$\begin{aligned} & 1 + 2c^2 + \cdots \\ =\ & \left(1 + 2a^2 + \cdots\right)\left(1 + 2b^2 + \cdots\right) - (2a + \cdots)(2b + \cdots)\cos\theta \\ =\ & 1 + 2a^2 + 2b^2 + \cdots - (4ab + \cdots)\cos\theta \end{aligned}$$

Thus

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

ignoring terms of higher degree, which will be negligible for small values of a, b, c.

Writing the hyperbolic law of cosines as

$$\cos\theta = \coth 2a \coth 2b - \operatorname{csch} 2a \operatorname{csch} 2b \cosh 2c$$

we can determine the angles at P and Q without further use of Hermitian matrices. Let ϕ be the angle at P and ψ be the angle at Q. Then

$$\begin{aligned}\cos\phi &= \coth 2a \coth 2c - \operatorname{csch} 2a \operatorname{csch} 2c \cosh 2b \\ \cos\psi &= \coth 2b \coth 2c - \operatorname{csch} 2b \operatorname{csch} 2c \cosh 2a\end{aligned}$$

Let $A = \cosh 2a, B = \cosh 2b, C = \cosh 2c$, and let $R = \sqrt{1 + 2ABC - A^2 - B^2 - C^2}$. Then

$$\begin{aligned}\sin\theta &= (\operatorname{csch} 2a)(\operatorname{csch} 2b) R \\ \sin\phi &= (\operatorname{csch} 2a)(\operatorname{csch} 2c) R \\ \sin\psi &= (\operatorname{csch} 2b)(\operatorname{csch} 2c) R\end{aligned}$$

and so

$$\begin{aligned}&\cos(\theta + \phi + \psi) \\ = \quad & -\frac{A + B + C + A^2 + B^2 + C^2 + AB + BC + CA - ABC}{(A+1)(B+1)(C+1)}\end{aligned}$$

which will be > -1 if and only if

$$A^2 + B^2 + C^2 < 1 + 2ABC.$$

This last condition holds if and only if

$$\begin{aligned}a &< b + c \\ b &< a + c \\ c &< a + b\end{aligned}$$

which is precisely the triangle inequality for a non-degenerate triangle. It follows that for a triangle in the hyperbolic plane, the sum of the angles is strictly less than π.

Lobachevsky's Formula: *The tangent of an acute angle in a right triangle is*

$$\frac{\tanh(2\cdot \text{opposite})}{\sinh(2\cdot \text{adjacent})}$$

.

Proof. For the angle ϕ between the sides of length a and c and with $R = \sqrt{1+2ABC-A^2-B^2-C^2}$ we have

$$\begin{aligned}
\cos\phi &= \coth 2a \coth 2c - \operatorname{csch} 2a \operatorname{csch} 2c \cosh 2b \\
\sinh 2a \sinh 2c \cos\phi &= AC - B \\
\sinh 2a \sinh 2c \sin\phi &= R
\end{aligned}$$

so that

$$\tan\phi = \frac{R}{AC-B}$$

Now assume $\theta = \frac{\pi}{2}$ so that $C = AB$ and $R^2 = \left(1-A^2\right)\left(1-B^2\right)$. Then

$$\begin{aligned}
\tan^2\phi &= \frac{\left(1-A^2\right)\left(1-B^2\right)}{B^2\left(A^2-1\right)^2} \\
&= \frac{\sinh^2 2b}{\cosh^2 2b \sinh^2 2a} \\
&= \frac{\tanh^2 2b}{\sinh^2 2a}
\end{aligned}$$

This is the hyperbolic analogue of the tangent of an acute angle in a Euclidean right triangle. As a corollary to Lobachevsky's formula, note that $\tan\phi \to \operatorname{csch} 2a$ as $b \to \infty$, that is, ϕ approaches the **angle of parallelism** as the right triangle approaches a singly asymptotic right triangle.

Exercise. Consider the right triangle OPQ with $OP = a, OQ = b$ and angle $\frac{\pi}{2}$ at O. Show that the line through P and Q is

$$H = \begin{pmatrix} 1 & -\coth 2a + i\coth 2b \\ -\coth 2a - i\coth 2b & 1 \end{pmatrix}.$$

Sine Formula: *The sine of an acute angle in a right triangle is*

$$\frac{\sinh(2\cdot \text{opposite})}{\sinh(2\cdot \text{hypotenuse})}$$

Proof. Again from $\tan\phi = \frac{R}{AC-B}$ we have in general that $\frac{\tan^2\phi}{1+\tan^2\phi} = \sin^2\phi = \left(B^2-2ABC+R^2+A^2C^2\right)^{-1} R^2$

If $C = AB$ then

$$\begin{aligned}\sin^2\phi &= \frac{R^2}{(A^2-1)(C^2-1)}\\ &= \frac{B^2-1}{C^2-1}\\ &= \frac{\sinh^2 2b}{\sinh^2 2c}\end{aligned}$$

This is the hyperbolic analogue of the tangent of an acute angle in a Euclidean right triangle.
Exercise. What happens to $\sin\phi$ as $b \to \infty$?

.

As an application of these results we can find a formula for the **Altitude of an Isosceles Triangle:**

Consider the triangle OAB with $OA = OB = s$ and vertex angle θ at O. As usual, let $O = 0$. This time we can simplify calculation by letting

$$\begin{aligned}A &= e^{i\frac{\theta}{2}}\tanh s\\ B &= e^{-i\frac{\theta}{2}}\tanh s\end{aligned}$$

Then OC is the length of the altitude from the vertex, where C is the intersection of the line through A and B with the real line. Since the line through A and B is

$$\begin{pmatrix}\cos\frac{\theta}{2} & -\coth 2s\\ -\coth 2s & \cos\frac{\theta}{2}\end{pmatrix}$$

or $\left(x - \sec\frac{\theta}{2}\coth 2s\right)^2 + y^2 = \left(\sec\frac{\theta}{2}\coth 2s\right)^2 - 1$, it follows that $C = \sec\frac{\theta}{2}\coth 2s - \sqrt{\sec^2\frac{\theta}{2}\coth^2 2s - 1}$.

Theorem. *The altitude of an isosceles triangle in $\mathcal{D}$ with vertex angle θ and side length s has length*

$$\begin{aligned}&\tanh^{-1}\left(\sec\frac{\theta}{2}\coth 2s - \sqrt{\sec^2\frac{\theta}{2}\coth^2 2s - 1}\right)\\ =\ &\frac{1}{4}\ln\frac{\coth 2s + \cos\frac{\theta}{2}}{\coth 2s - \cos\frac{\theta}{2}}\end{aligned}$$

Note that $\lim_{s\to\infty} C = \sec\frac{\theta}{2} - \tan\frac{\theta}{2}$ (and that $\lim_{s\to\infty}\tanh^{-1} C = \frac{1}{2}\ln\cot\frac{\theta}{4}$), so the altitude through the vertex of an isosceles triangle is bounded by the vertex angle. For example, if $\theta = \frac{\pi}{2}$ then $C = \sqrt{2}\coth 2s - \sqrt{2\coth^2 2s - 1}$ and so the maximum length of the altitude of an isosceles right triangle is $\tanh^{-1}\left(\sqrt{2}-1\right)$, the distance between a right curve and its axis. Note also that if the isosceles triangle is equilateral then s and θ determine each other so the length of any altitude is a function of either one.

Exercise. What is the upper bound in the equilateral case?

Equilateral Triangles

Let POQ be an equilateral triangle with vertex angle θ and side length s. We can assume $P = \tanh s, O = 0, Q = e^{i\theta}\tanh s$. By the **Sine Formula**

$$\begin{aligned} \sin\frac{\theta}{2} &= \frac{\sinh s}{\sinh 2s} \\ &= \frac{1}{2}\operatorname{sech} s \end{aligned}$$

Then

$$\begin{aligned} \tanh^2 s &= 1 - 4\sin^2\frac{\theta}{2} \\ &= 2\cos\theta - 1 \end{aligned}$$

It follows that, in polar coordinates, the locus of Q is

$$(1) \qquad r = \sqrt{2\cos\theta - 1}$$

In particular, $\theta \to 0$ as $s \to \infty$ and $\theta \to \frac{\pi}{3}$ as $s \to 0$.

As a corollary we find that the altitude h of an equilateral triangle can be expressed in terms of its side length s or interior angle θ as

$$\begin{aligned} h &= \frac{1}{4}\ln\frac{\cosh 2s + \sqrt{\left(\cosh^2 2s - \cosh^2 s\right)}}{\cosh 2s - \sqrt{\left(\cosh^2 2s - \cosh^2 s\right)}} \\ &= \frac{1}{2}\tanh^{-1}\frac{\sinh s}{1 + 2\sinh^2 s}\sqrt{3 + 4\sinh^2 s} \\ &= \frac{1}{2}\tanh^{-1}\frac{\sqrt{(2\cos\theta - 1)(\cos\theta + 1)}}{\sqrt{2}\cos\theta}. \end{aligned}$$

As an application, we can let the side length, and therefor the interior angle, of an equilateral triangle vary and look at the collection of triangles with fixed vertex and side on a given line. For example, consider the locus of circumcenters for the equilateral triangles POQ as P varies on the real line. The $\perp$-bisector of side PQ is the line $y = x \tan \frac{\theta}{2} = x \frac{\sin\theta}{1+\cos\theta}$, so the circumcenter is the intersection of this line with the $\perp$-bisector of side OP, which is the line $\begin{pmatrix} 1 & -\coth s \\ -\coth s & 1 \end{pmatrix}$ whose Cartesian form is

$$(x - \coth s)^2 + y^2 = \operatorname{csch}^2 s$$

Using $\tanh^2 s = 2\cos\theta - 1$ from (1) together with $y^2 = x^2 \frac{1-\cos\theta}{1+\cos\theta}$, the intersection in $\mathcal{D}$ is found to be

$$\begin{aligned} x &= \sqrt{\frac{\cos\theta - \sqrt{3}\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}}\cos\theta \\ y &= \sqrt{\frac{\cos\theta - \sqrt{3}\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}}\sin\theta \end{aligned} \tag{2}$$

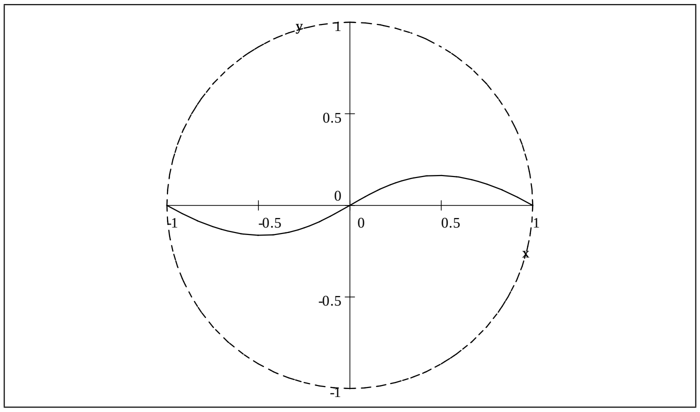
This gives the locus of circumcenters parametrically in terms of θ, from which the polar form of the curve is

$$r = \sqrt{\frac{\cos\theta - \sqrt{3}\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}} \tag{3}$$

It is now straightforward to put (3) into Cartesian form

$$\left(x^2 + y^2\right)\left(x + \sqrt{3}y\right) = x - \sqrt{3}y \tag{4}$$

Equation (4) describes the real points of a non-singular cubic, an example of an elliptic curve, which is an algebraic curve of genus 1:



This locus will be an elliptic curve for any fixed vertex and any given line because any Mobius transformation induces a transformation on algebraic curves that preserves genus. Consequently this locus will be an elliptic curve in any inversive model of the hyperbolic plane.

Arc Length in $\mathcal{D}$

In $\mathcal{D}$, the distance between the points z and z_1 is given by

$$s(z, z_1) = \tanh^{-1}\left|\frac{z - z_1}{1 - zz_1^*}\right|$$

from which we can obtain an arc length differential by the following informal argument (justified by the Mean Value Theorem). For z_1 very close to z we replace $z - z_1$ with dz and $1 - zz_1^*$ by $1 - |z|^2$, which is a positive real number. Let $s = s(z, z_1)$ and $u = \left|\frac{z-z_1}{1-zz_1^*}\right|$

$$\begin{aligned} du &= \frac{|dz|}{1 - |z|^2} \\ &= \frac{ds}{\cosh^2 s} \end{aligned}$$

and so $ds = \frac{|dz|}{1-|z|^2}\cosh^2 s$, which can be replaced with

$$ds = \frac{|dz|}{1 - |z|^2}$$

for purposes of integration since $\cosh^2 s = 1 + s^2 + \cdots \approx 1$ on the scale of ds. Here, $dz = dx + idy$ and so $|dz| = \sqrt{(dx)^2 + (dy)^2}$, the Riemannian form typically used to compute the length of a path in $\mathcal{D}$. For example, the circle C in $\mathcal{D}$ centered at 0 with hyperbolic radius k is described by the Cartesian equation

$$x^2 + y^2 = \tanh^2 k$$

which can be expressed parametrically as

$$\gamma(\theta) = (\tanh k \cos\theta, \tanh k \sin\theta)$$

and so

$$\gamma'(\theta) = (-\tanh k \sin\theta, \tanh k \cos\theta).$$

The circumference of C is the integral

$$\begin{aligned} &\int_C ds \\ = &\int_C \frac{|dz|}{1 - |z|^2} \end{aligned}$$

where $|z| = |\gamma(\theta)| = \tanh k$, and $|dz| =$

$$\sqrt{(-\tanh k \sin\theta d\theta)^2 + (\tanh k \cos\theta d\theta)^2} = \tanh k d\theta$$

Then

$$\begin{aligned} & \frac{|dz|}{1-|z|^2} \\ = & \frac{\tanh k}{1-\tanh^2 k} d\theta \\ = & \frac{1}{2}\sinh 2k d\theta \end{aligned}$$

and so the circumference is

$$\frac{1}{2}\int_0^{2\pi} \sinh 2k d\theta = \pi \sinh 2k$$

Since $\lim\limits_{k\to 0} \frac{\pi\sinh 2k}{2\pi k} = 1$ we have the usual result that the hyperbolic measure agrees with the Euclidean measure in the infinitesimal limit. Of greater interest is the fact that

$$\frac{\pi\sinh 2k}{2k}$$

the ratio of hyperbolic circumference to hyperbolic diameter is not constant and is always $> \pi$ (though it does approach π as $k \to 0$). Note also that this ratio is unbounded as $k \to \infty$.

In general, arc length integrals can be difficult to evaluate in exact form although paths along inversive lines are usually manageable. As an example, let

$$\begin{aligned} \gamma(\theta) &= \sqrt{2}\sin\theta + i\left(\sqrt{2}\cos\theta - 1\right) \\ -\frac{\pi}{4} &< \theta < \frac{\pi}{4} \end{aligned}$$

This path is a right curve for the real line. Let $z_\alpha = \gamma(\alpha)$. Then $z_0 = i\left(\sqrt{2}-1\right)$ we can measure the distance along this curve between the imaginary axis and the point z_α for $0 \le \alpha < \frac{\pi}{4}$ by evaluating the integral

$$\begin{aligned} & \int_0^\alpha \frac{d\theta}{2\cos\theta - \sqrt{2}} \\ = & \frac{\sqrt{2}}{4}\ln\frac{1+\sin\left(\alpha-\frac{\pi}{4}\right)}{1-\sin\left(\alpha+\frac{\pi}{4}\right)} \end{aligned}$$

Thus, the distance from z_0 to $z_{\frac{\pi}{6}} = \frac{1}{2}\left(\sqrt{2} + i\left(\sqrt{6}-2\right)\right)$ along the curve is $\frac{1}{\sqrt{2}}\ln\left(\frac{1}{2}\sqrt{2} + \sqrt{3} + \frac{1}{2}\sqrt{6} + 1\right) \approx 1.0888$. Note that $s\left(z_0, z_{\frac{\pi}{6}}\right) =$

$$\begin{aligned} & \tanh^{-1}\left|\frac{z_0 - z_{\frac{\pi}{6}}}{1 - z_0^* z_{\frac{\pi}{6}}}\right| \\ = & \tanh^{-1}\sqrt{\frac{1}{5}\sqrt{2} - \frac{1}{5}\sqrt{2}\sqrt{6} + 1} \\ \approx & 1.0157 \end{aligned}$$

so the distance along the curve is, in fact, greater than the length of the line segment between these points.

Arc Length in $\mathcal{U}$

We can obtain an arc length measure in the half-plane model $\mathcal{U}$, as follows. First, the distance between two points in $\mathcal{U}$ can be found by mapping the points to $\mathcal{D}$, by $T(z) = \frac{z-i}{1-iz}$ for example, and computing the distance as above. (It actually does not matter which LFT is used in mapping $\mathcal{U}$ to $\mathcal{D}$; see Second Project Assignment **4.**) The resulting metrics will differ by at most a constant factor. Let $w = \frac{z-i}{1-iz}$. Then the arc length differential in $\mathcal{U}$ should take the form

$$ds = \frac{|dw|}{1 - |w|^2}$$

and this measure will be invariant under the collineation group G_U. This group contains the Euclidean subgroup of translations $z \mapsto z + b, b \in \mathbb{R}$, which is a conjugate in the Möbius group of the group of rotations about 0 in $\mathcal{D}$ (see **Exercise 2)**, below). Thus, we can get information about how ds measures length in $\mathcal{U}$ by restricting w to be on the imaginary line $z = it, t > 0$. In this case

$$\begin{aligned} w &= i\frac{t-1}{t+1} \\ |dw| &= \frac{2}{(1+t)^2}\,|dt| \\ 1 - |w|^2 &= \frac{4t}{(1+t)^2} \end{aligned}$$

It follows that $ds = \frac{1}{2t}\,|dt|$, although this is usually normalized to

$$ds = \frac{1}{t}\,|dt|$$

The important information for the general situation is that the denominator represents the imaginary part of z, so the arc length differential in $\mathcal{U}$ is

$$\begin{aligned} ds &= \frac{|dz|}{\mathrm{Im}(z)} \\ &= \frac{\sqrt{(dx)^2 + (dy)^2}}{y} \end{aligned}$$

As an example, consider the path $y = x$ in $\mathcal{U}$, which is right curve for the imaginary line. The simplest smooth parameterization of this path is

$$\begin{aligned} x(t) &= t, y(t) = t \\ dx &= dt = dy \\ \sqrt{(dx)^2 + (dy)^2} &= \sqrt{2}\,|dt| \\ &= \sqrt{2}dt \end{aligned}$$

for increasing t. Thus the length of the path between the points $a + ai$ and $b + bi, b > a$, is

$$\begin{aligned} & \sqrt{2}\int_a^b \frac{dt}{t} \\ = & \sqrt{2}\ln\frac{b}{a} \end{aligned}$$

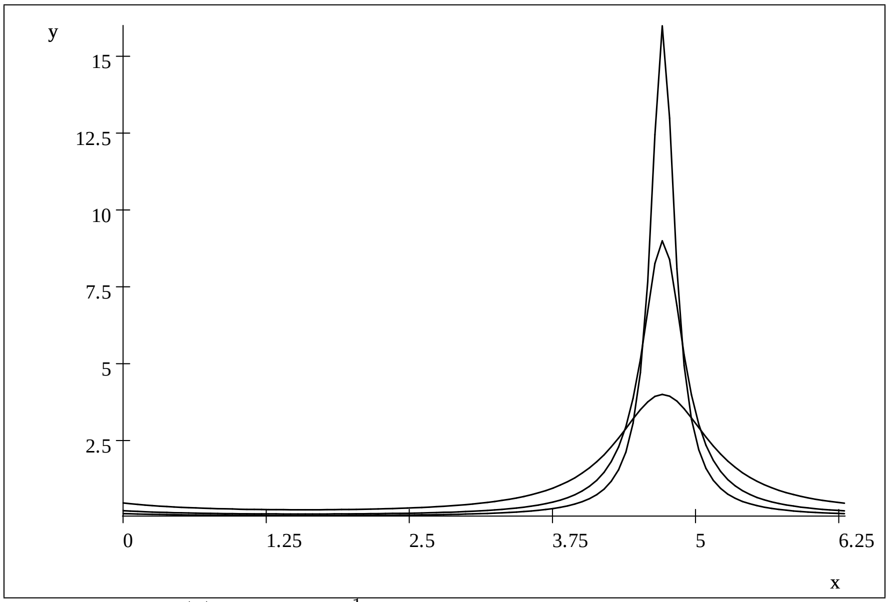
Now consider a circle in $\mathcal{U}$ of radius k. If the circle is centered at i then, see **Exercise 3**, its inversive center is at $i\cosh 2k$ and so $\gamma(\theta) = \sinh 2k\cos\theta + i(\sinh 2k\sin\theta + \cosh 2k)$. The circumference of the circle in $\mathcal{U}$ is

$$\int\limits_0^{2\pi} \frac{\sin 2k d\theta}{(\sinh 2k)\sin\theta + \cosh 2k}$$

however

$$\int\limits_0^{2\pi} \frac{d\theta}{(\sinh 2k)\sin\theta + \cosh 2k} = 2\pi$$

independent of k, and so the circumference is $2\pi\sinh 2k$, which agrees with the result in $\mathcal{D}$ because we normalized the arc length differential.



$f(\theta) = \frac{1}{(\sinh 2k)\sin\theta + \cosh 2k}$ for various values of k

Exercise 1) Show that $\sqrt{2}\ln\frac{b}{a}$ is greater than the hyperbolic distance between $a+ai$ and $b+bi$, either by mapping them to $\mathcal{D}$ or by deriving the corresponding distance formula in $\mathcal{U}$.

Exercise 2) Consider the group of LFTs $G = \{z \mapsto z + b : b \in \mathbb{R}\}$. Show that G is a conjugate in $\mathcal{M}$ of the group of LFTs $G' = \{z \mapsto e^{i\theta}z : \theta \in [0, 2\pi)\}$.

Exercise 3) Show that the circle in U with center at i and radius k has inversive center at $i\cosh 2k$ and inversive radius $\sinh 2k$.

Area in $\mathcal{D}$

If Ω is a region in $\mathcal{D}$ bounded by simple closed curves, we want $\mathcal{A}(\Omega)$ to be measured by

$$\iint\limits_{\Omega} f(z)dxdy$$

where f is a positive and differentiable real-valued function of x and y. Then the area measure will have the usual additivity property. To ensure that it also has the property of invariance under G_D, first note that if $z = x + iy$ and $w = T(z) = \frac{az+b}{cz+d} = u + iv$ then

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{|ad - bc|^2}{|cz + d|^4}$$

In particular, if $T_m = \begin{bmatrix} 1 & -m \\ -\overline{m} & 1 \end{bmatrix}$ then this Jacobian is

$$J(z) = \frac{\left(1 - |m|^2\right)^2}{|1 - m^* z|^4}$$

Since the canonical form of a direct collineation is T_m for some m with $|m| < 1$ followed by rotation about 0, it suffices to find a function f so that the above integral is invariant under T_m, rotation about 0, and complex conjugation. For $T \in G_D$, let $\tilde{\Omega} = T(\Omega)$. Then, by the change-of-variable formula for plane integrals, invariance of $\mathcal{A}$ becomes

$$\begin{aligned} \mathcal{A}\left(\tilde{\Omega}\right) &= \iint\limits_{\tilde{\Omega}} f(w)dudv \\ &= \iint\limits_{\Omega} f(T(z))J(z)dxdy \\ &= \iint\limits_{\Omega} f(z)dxdy = \mathcal{A}(\Omega) \end{aligned}$$

This invariance condition, together with the Mean Value Theorem for integrals, yields the so-called *functional equation* for f

$$f(z) = f(T(z))J(z).$$

If $m = t \in \mathbb{R}$ and T_t is restricted to the real line then $w = u = \frac{x-t}{1-tx}$, and the functional equation becomes

$$f(x)(1 - tx)^4 = f(u)\left(1 - t^2\right)^2$$

Since this identity must hold for all real t it can be differentiated with respect to t

$$\begin{aligned}-4x\,(1-tx)^3 f(x) &= f'(u)u'(t)\left(1-t^2\right)^2 - 4t\left(1-t^2\right) f(u)\\ &= f'(u)\frac{x^2-1}{(1-tx)^2}\left(1-t^2\right)^2 - 4t\left(1-t^2\right) f(u)\end{aligned}$$

and then evaluated at $t=0$ to obtain

$$-4xf(x) = f'(x)\left(x^2-1\right)$$

a first-order linear differential equation whose general solution is

$$f(x) = \frac{k}{\left(1-x^2\right)^2}$$

where k is a positive constant since f is positive on $\mathcal{D}$. By convention, we choose to let $k=1$. Rotational invariance requires $f(z) = f\left(|z|\right) = \frac{1}{\left(1-|z|^2\right)^2}$, which is also invariant under complex conjugation. Thus

$$\begin{aligned}\mathcal{A}\left(\Omega\right) &= \iint\limits_{\Omega} \frac{1}{\left(1-x^2-y^2\right)^2} dxdy\\ &= \iint\limits_{\Omega} \frac{r}{\left(1-r^2\right)^2} drd\theta\end{aligned}$$

in both rectangular and polar form. As an example, let Ω be a region bounded by a circle of radius s in $\mathcal{D}$, which we can take to be the region bounded by the circle $z = (\tanh s)\, e^{i\theta}$. Then

$$\begin{aligned}\mathcal{A}\left(\Omega\right) &= \int_0^{2\pi}\int_0^{\tanh s} \frac{r}{\left(1-r^2\right)^2} drd\theta\\ &= \pi\frac{1}{1-r^2}\Big|_0^{\tanh s}\\ &= \pi\sinh^2 s.\end{aligned}$$

Since $\lim\limits_{s\to 0}\frac{\pi\sinh^2 s}{\pi s^2} = 1$ the formula for area of a circular region in $\mathcal{D}$ agrees with Euclidean measure in the infinitesimal limit.

The area of a triangle in $\mathcal{D}$ can be computed as follows. It suffices to find the area of a right triangle since the additivity property can be used after partitioning any triangle into two right triangles by dropping an altitude. So, let Ω be the region with triangular boundary OAB where $O = 0$ and $A = \tanh a$. Place the right angle at A and let ϕ be the angle at O, ψ the angle at B. Let $z = r(\theta) e^{i\theta}, 0 \leq \theta \leq \phi$, be a point on side AB. Then

$$\mathcal{A}(\Omega) = \int_0^{\phi} \int_0^{r(\theta)} \frac{r}{(1-r^2)^2} dr d\theta$$

and since $r(\theta) = \tanh s(\theta)$, where $s(\theta) = s(0, z)$, we have $dr = \left(\operatorname{sech}^2 s\right) ds$ and so the integral can be written

$$\begin{aligned} \mathcal{A}(\Omega) &= \int_0^{\phi} \int_0^{s(\theta)} (\sinh s \cosh s)\, ds d\theta \\ &= \frac{1}{4} \int_0^{\phi} (\cosh 2s(\theta) - 1)\, d\theta \end{aligned}$$

From Lobachevsky's Formula and/or the Sine Formula it follows that $\tanh 2a = \cos\theta \tanh 2s(\theta)$ and then $\cosh 2s(\theta) = \frac{\cos\theta}{\sqrt{\cos^2\theta - \tanh^2 2a}}$, and so

$$4\mathcal{A}(\Omega) = \int_0^{\phi} \left(\frac{\cos\theta}{\sqrt{\cos^2\theta - \tanh^2 2a}} - 1 \right) d\theta$$

Now let $u = \sin\theta$ so that $du = \cos\theta d\theta$ and $\cos^2\theta = 1 - u^2$. Then

$$\begin{aligned} 4\mathcal{A}(\Omega) &= \int_0^{\sin\phi} \left(\frac{du}{\sqrt{\operatorname{sech}^2 2a - u^2}} \right) - \phi \\ &= \sin^{-1}(\sin\phi \cosh 2a) - \phi \end{aligned}$$

since $\sin^{-1}\frac{u}{b}$ is an anti-derivative for $\frac{1}{\sqrt{b^2-u^2}}$. Finally, from the hyperbolic Pythagorean Theorem it follows that

$$\sin\phi \cosh 2a = \cos\psi$$

because $\cosh 2a \cosh 2AB = \cosh 2OB$, whereas $\sin\phi = \frac{\sinh 2AB}{\sinh 2OB}$ and $\sin\psi = \frac{\sinh 2a}{\sinh 2OB}$. Thus

$$\begin{aligned} 4\mathcal{A}(\Omega) &= \sin^{-1}(\cos\psi) - \phi \\ &= \left(\frac{\pi}{2} - \psi\right) - \phi \\ &= \pi - \left(\frac{\pi}{2} + \psi + \phi\right) \end{aligned}$$

and so the area of a right triangle is a constant multiple of the difference between π and the sum of its interior angles. It follows by additivity that if Ω is a triangular region with interior angles α, β, γ then $\delta(\Omega) = \pi - (\alpha + \beta + \gamma)$ is a constant multiple of $\mathcal{A}(\Omega)$. By adjusting the constant of integration in ψ so that $\delta(\Omega) = \mathcal{A}(\Omega)$ we can state this result as Gauss did: *The area of a triangle is equal to its angular defect.* Note also that this result is ordered: If triangular region Υ is properly contained in Ω then $\mathcal{A}(\Upsilon) < \mathcal{A}(\Omega)$, because the sum of angles in Υ will be greater than the sum of angles in Ω. Using improper Riemann integration it is easy to extend the result to asymptotic triangles. In particular, an asymptotic triangle has finite area and the area of any triply-asymptotic triangle is π.

Tesselations

A *regular tesselation* is a dissection of the plane into regular polygons. If all the polygons are congruent then the tesselation is called *uniform*. Let $\{p, q\}$ denote the uniform regular tesselation by p-gons, q of them at each vertex. Then the angle at each vertex is $\frac{2\pi}{q}$ and so the angle sum within each p-gon is $2\pi\frac{p}{q}$. In the Euclidean plane the angle sum within a p-gon is $(p-2)\,\pi$; setting

$$2\pi\frac{p}{q} = (p-2)\,\pi$$

yields

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$$

and so the only uniform regular tesselations of the Euclidean plane are $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$.

In the hyperbolic plane, the angle sum within a p-gon can be any number less than $(p-2)\,\pi$; it follows that the uniform regular tesselation $\{p, q\}$ exists provided

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$$

For example, a tesselation by equilateral triangles is possible with q triangles at a vertex provided $q \geq 7$. We have seen that the vertex angle θ of an equilateral triangle determines the side length s according to the relation

$$\tanh s = \sqrt{2\cos\theta - 1}$$

For the $\{3, q\}$ tesselation the angle at each vertex is $\frac{2\pi}{q}$, and so

$$\tanh s = \sqrt{2\cos\frac{2\pi}{q} - 1}.$$

For example, if $q = 8$ then the angle at each vertex is $\frac{\pi}{4}$, and $s = \tanh^{-1}\sqrt{\sqrt{2} - 1}$. Since $\tanh^{-1}\sqrt{2\cos\frac{2\pi}{q} - 1}$ is an increasing function of q the smallest possible side length for a tessellation by equilateral triangles occurs for the $\{3, 7\}$ tesselation, where $s = \tanh^{-1}\sqrt{2\cos\frac{2\pi}{7} - 1} \approx 0.545\,27$. The *dual* of this tesselation, the $\{7, 3\}$ tesselation, is often depicted in books on hyperbolic geometry (see the cover of *Geometry*, by Brannan/Esplen/Gray, Cambridge University Press). Note that for this tesselation the angle at each vertex is $\frac{2\pi}{3}$, so someone standing at a vertex might infer that the tiling is by hexagons if they did not know the space was hyperbolic.

Important Skills

Determining types of hyperbolic collineations (direct or opposite; if direct, rotation, limit rotation, or translation)
Collineations as products of reflections
Relation of center and radius of a circle to its inversive center and radius
Determining perpendiculars and midpoints of segments

Determining parallels and angle of parallelism
Equidistant curves to a given line
Horocycles at a boundary point

Law of Cosines and applications to right triangles (Lobachevsky's Theorem, Sine Formula)
Arc length along parameterized paths
Area of regions, defect of a triangle