MATH 529-01: Advanced Geometry (82616)
JB-387, MW 4 -5:50PM
SYLLABUS Fall 2011

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Text: Brannan/Esplen/Gray
Geometry (Cambridge University Press)

Prerequisites: MATH 329, MATH 331 and MATH 345

The main goal of MATH 529 is to study the geometry of the real projective plane and the associated classical theorems from the viewpoint of projective transformations. This approach is sometimes referred to as the Kleinian viewpoint, after Felix Klein, the architect of the Erlangen Programme. This blueprint for classifying geometries was realized in the last quarter of the nineteenth century and we now use it in the language of group theory to obtain the principal theorems of geometry in an efficient manner.

In this text the construction of the projective plane is accomplished in **Chapter 3**. In **Chapter 4** we apply this construction to the study of conics. The material of **Chapter 2** provides the necessary background from the theory of affine transformations. We will take sufficient time to review this material. Effective study of this material requires careful reading of the text, including working through the **problems** embedded in the text (most of which are solved in **Appendix 3**) as well as the **exercises** at the end of each chapter.

Grading will be based on two graded assignments (**10% first, 20% second**), a midterm exam (**30%**) and the final exam (**40%**). The second assignment must be presented in a typed format and may be submitted electronically. For the in-class exams you should bring your notes since the purpose is to learn how to use your own work when processing the material.

After computing your total scores weighted according to these percentages, course grades will be assigned as follows:

	$A :\geq 91$	$A- : 86-90$
$B+ : 81-85$	$B : 76-80$	$B- : 71-75$
$C+ : 66-70$	$C : 61-65$	$C- : 56-60$
$D : 45-55$	$F :< 45$	

Some important dates:

September 26 - First day of class
September 28 - Last day to add classes over MyCoyote for Fall quarter
October 12 - Fall CENSUS; last day to submit add/drop slips
October 24 - Winter Advising begins
October 26 - Midterm exam
October 31 - Winter Priority Registration begins
November 30 - Last day of class
December 7 - Final Exam

Schedule of Topics:

Week 1
Affine transformations and conics

Week 2
Introduction to projective space

Week 3
Projective transformations; Fundamental Theorem of Projective Geometry

Week 4
Theorems of Desargues, Pappus, Brianchon; Cross-ratio

Week 5
Projective conics; tangents and secants; midterm exam

Week 6
Relation to affine conics; Joachimsthal notation

Week 7
Poles and polars; LaHire's Theorem

Week 8
Standard forms; determination of conics

Week 9
Theorems on tangents and secants

Week 10
Pascal's Theorem, its dual and converse

Affine Conics and Degenerate Cases

Let A, B, C, D, E be real numbers, not all zero. Then the Cartesian equation $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ represents the general real conic through $(0,0)$ in $\mathbb{R}^2$. In particular, this conic is not empty and we consider it to be *degenerate* if it consists of either **1)** a single point; **2)** a single line or a pair of parallel lines; **3)** two intersecting lines. The equation can be expressed in the matrix form

$$\begin{pmatrix} x & y \end{pmatrix} \left[Q \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D \\ E \end{pmatrix} \right] = 0$$

where $Q = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$. The degenerate cases can be determined as follows. First, suppose $D = E = 0$. The conic $Ax^2 + Bxy + Cy^2 = 0$ is easily seen to be degenerate because the equation is equivalent to

$$2Cy = \left(-B \pm \sqrt{B^2 - 4AC} \right) x$$

which presents three cases:

If $B^2 - 4AC < 0$ then the only real solution is the single point $(x, y) = (0, 0)$.

If $B^2 - 4AC = 0$ then $Bx + 2Cy = 0$, which is a single line provided at least one of B, C is non-zero; if $B = C = 0$ then $A \neq 0$ and so the conic is the line $x = 0$.

Finally, if $B^2 - 4AC > 0$ then $B \neq 0$ and we have two lines through the origin provided $C \neq 0$; but if $C = 0$ then the conic is $x\,(Ax + By) = 0$ and so the two lines are $x = 0$ and $Ax + By = 0$.

Note that $\det Q = AC - \frac{1}{4}B^2$ and so the three degenerate cases here correspond, respectively, to $\det Q > 0$, $\det Q = 0$, $\det Q < 0$. In the non-degenerate situation this is precisely the discriminant test for an affine conic, so we consider a single point to be a *degenerate ellipse*, a single line to be a *degenerate parabola*, and two intersecting lines to be a *degenerate hyperbola*. If at least one of D, E is non-zero we can usually apply a translation to make them both vanish, but not always since we do not obtain the case of a pair of parallel lines from $Ax^2 + Bxy + Cy^2 = 0$. Therefore it is worth looking at the general case, which, in particular, will suggest that a pair of parallel lines is a degenerate parabola.

1) Degenerate ellipse

If $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is the single point $(0,0)$ then $x = 0$ if and only if $y = 0$. Thus neither linear equation

$$\begin{aligned} Cy + E &= 0 \\ Ax + D &= 0 \end{aligned}$$

can have a non-zero solution. If $Cy + E = 0$ has only $y = 0$ as a solution then $E = 0$ and $C \neq 0$, and if $Ax + D = 0$ has only $x = 0$ as a solution then $D = 0$

and $A \neq 0$. It follows that the conic is

$$Ax^2 + Bxy + Cy^2 = 0$$

where $Ax^2+Bxy+Cy^2$ is a *definite* quadratic form, i.e., $\det Q = AC - \frac{1}{4}B^2 > 0$, in agreement with our analysis above. *We conclude that $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a single point provided $D = E = 0$ and $4AC > B^2$. In particular, both A and C are non-zero.*

2) Degenerate parabola

If $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a single line through $(0, 0)$ then either $Ax^2 + Bxy + Cy^2 + Dx + Ey = \alpha x + \beta y$ or $Ax^2 + Bxy + Cy^2 + Dx + Ey = \pm (\alpha x + \beta y)^2$, for some choice of α, β not both 0. The first case implies $A = B = C = 0$ and at least one of D, E non-zero. The second case can only happen if $D = E = 0$, whereby $Ax^2 + Bxy + Cy^2 = \pm \left(\alpha^2 x^2 + 2\alpha\beta xy + \beta^2 y^2\right)$ so $4AC = B^2$; conversely, the conditions $D = E = 0$ and $\det Q = 0$ imply the conic is a single line. To see this, note that if either $A = 0$ or $C = 0$ then also $B = 0$ (because $4AC = B^2$) so the conic is one of the axes in this case. Assume then that $A \neq 0$ and $C \neq 0$. Since $AC > 0$ we can assume both are positive and we have $Ax^2+Bxy+Cy^2 = Ax^2 \pm 2\sqrt{AC}xy + Cy^2 = \left(\sqrt{A}x \pm \sqrt{C}y\right)^2$, resulting in the single line $\sqrt{A}x \pm \sqrt{C}y = 0$. Thus $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a single line provided $A = B = C = 0$ and at least one of D, E non-zero, or, $D = E = 0$ and $4AC = B^2$. If $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a a pair of parallel lines then $Ax^2 + Bxy + Cy^2 + Dx + Ey = \pm (\alpha x + \beta y)(\alpha x + \beta y + \nu) = \pm \left(\alpha^2 x^2 + 2\alpha\beta xy + \beta^2 y^2 + \alpha\nu x + \beta\nu y\right)$, for some choice of α, β not both 0 and $\nu \neq 0$. This is equivalent to the condition that (A, B, C) be a non-zero multiple of$\left(D^2, 2DE, E^2\right)$ with at least one of D, E non-zero. In particular, $4AC = B^2$. *We conclude that $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is*

a) *a single line if $A = B = C = 0$, or if $D = E = 0$ and $4AC = B^2$;*

b) *a pair of parallel lines if at least one of D, E is non-zero and (A, B, C) is a non-zero multiple of $\left(D^2, 2DE, E^2\right)$.*

In either case $4AC = B^2$.

3) Degenerate hyperbola

If $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a pair of intersecting lines then at least one of them passes through $(0, 0)$ and so $Ax^2 + Bxy + Cy^2 + Dx + Ey = (\alpha x + \beta y)(\lambda x + \mu y + \nu)$ with $\beta\lambda \neq \alpha\mu$. In particular, $4AC - B^2 = -(\alpha\mu - \beta\lambda)^2 < 0$, and, though not immediately obvious, it also follows that $AE^2 - BDE + CD^2 = 0$. In fact, these two conditions are also sufficient for the conic to be a pair of intersecting lines since they allow $\alpha, \beta, \lambda, \mu, \nu$ to be found in terms of A, B, C, D, E. *We conclude that $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is a pair of intersecting lines if $AE^2 - BDE + CD^2 = 0$ and $4AC - B^2 =< 0$.*

Note that $AE^2 - BDE + CD^2 = 0$ holds for each of the degenerate cases (check that this is true for case **2b**). This condition is also sufficient for the conic to be degenerate.

Theorem. Let $\boldsymbol{Q} = \begin{pmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 0 \end{pmatrix}$. The conic $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is degenerate if and only if $\det \boldsymbol{Q} = 0$.
We will prove this theorem, in greater generality, when we study conics in the projective plane.

Eccentricity of Affine Conics

The eccentricity ε of an affine conic is a parameter that was originally derived in terms of the intersection of a plane with a cone. For example, if the cone is a surface of revolution obtained by rotating a line (generator) about an axis line and Π is a plane perpendicular to this axis then

$$\varepsilon = \frac{\sin\alpha}{\sin\beta}$$

where α is the angle between Π and the intersecting plane and β is the angle between Π and the generator of the cone. In particular, $\varepsilon = 1$ when $\alpha = \beta$. Note that $\csc\beta$ is an upper bound for ε, so to obtain hyperbolas of arbitrarily large eccentricity it is necessary to "fatten" the cone. From the viewpoint of planar affine geometry this definition is not intrinsic because it requires an extra dimension. A definition in terms of Q that agrees with the three-dimensional construction, and that unifies the definitions of eccentricity used in calculus, is provided by linear algebra. Since Q is a real symmetric matrix the roots of its characteristic polynomial are real. Let these eigenvalues be λ and μ. Then $\det Q = \lambda\mu$ and so the conic is an ellipse (hyperbola) if λ, μ have the same (opposite) sign. If the conic is a parabola then, unless $A = B = C = 0$, exactly one of λ, μ is zero. If $\det Q \geq 0$, order the eigenvalues so that $|\lambda| \leq |\mu|$. Then

$$\varepsilon = \sqrt{1 - \frac{\lambda}{\mu}}$$

In particular, if $\det Q = 0$ then $\varepsilon = 1$. (We define ε to be 1 if $A = B = C = 0$.) If $\det Q < 0$ the situation is slightly more complicated because hyperbolas occur in conjugate pairs that share common asymptotes. For example, $x^2 - 3y^2 + 1 = 0$ has asymptotes $y = \pm\frac{1}{\sqrt{3}}x$ which it shares with $x^2 - 3y^2 - 1 = 0$. In this example, let $\lambda = 1$ and $\mu = -3$. Then the first hyperbola has eccentricity

$$\varepsilon = \frac{2}{\sqrt{3}} = \sqrt{1 - \frac{\lambda}{\mu}}$$

while the eccentricity of the second is

$$\varepsilon' = 2 = \sqrt{1 - \frac{\mu}{\lambda}}$$

The choice between $\frac{\lambda}{\mu}$ and $\frac{\mu}{\lambda}$ cannot be made from the quadratic form alone but

$$\left(\frac{\varepsilon}{\varepsilon'}\right)^2 = -\frac{\lambda}{\mu}$$

where $|\lambda| \leq |\mu|$ if $|\varepsilon| \leq |\varepsilon'|$. It is also straightforward to show that if θ is either of the two angles between the asymptotes then

$$|\tan\theta| = \frac{2\varepsilon\varepsilon'}{\left|\varepsilon^2 - (\varepsilon')^2\right|}$$

Conics Defined by Affine Transformations

Let T be a collineation of the Euclidean plane. A basic theorem tells us that T is an affine transformation, in particular it can be represented as a transformation of $\mathbb{R}^2$ in the form

$$T(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}, ad - bc \neq 0$$

Let P be any point that is not fixed by T and consider the pencil of lines L concurrent at P. We define $\mathcal{E}(T,P) = \{L \cap T(L) \mid P \in L\}$ to be the *conic at P afforded by T*. If $O = (0,0)$ then it is straightforward to show that $\mathcal{E} = \mathcal{E}(T,O)$ is the Cartesian curve

$$cx^2 + (d-a)xy - by^2 = (cp - aq)\,x + (dp - bq)\,y$$

whereby $\mathcal{E}$ is an ellipse if $\Delta > 0$, a parabola if $\Delta = 0$, an hyperbola if $\Delta < 0$; where $\Delta = 4\delta - \tau^2$, $\delta = ad - bc$ and $\tau = a + d$. Of course, this conic might be degenerate. For example, if

$$T(x,y) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then $\mathcal{E}(T,O)$ is the curve $y^2 - y = 0$, which consists of the the two parallel lines $y = 0$ and $y = 1$.

MATH 529-01: First Graded Assignment

Let T be the affine transformation defined by $T(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$ and let $\mathcal{E} = \{L \cap T(L) \mid O \in L\}$, where $O = (0,0)$. Then $\mathcal{E} = \mathcal{E}(T, O)$ is the Cartesian curve

$$cx^2 + (d-a)xy - by^2 = (cp - aq)\,x + (dp - bq)\,y$$

whereby $\mathcal{E}$ is an ellipse if $\Delta > 0$, a parabola if $\Delta = 0$, an hyperbola if $\Delta < 0$; where $\Delta = 4\delta - \tau^2$, $\delta = ad - bc$ and $\tau = a + d$. Present a complete solution to **one** of the following:

1. We say that T *preserves orientation* if $\delta > 0$ and *reverses orientation* if $\delta < 0$.

a) If T reverses orientation show that $\mathcal{E}$ is a hyperbola.

b) If T preserves orientation show that $\mathcal{E}$ can be any affine type.

c) If T is an isometry show that there are only two possibilities for $\mathcal{E}$ up to similarity. Give an example of each case.

2. Note that the eccentricity of $\mathcal{E}$ depends only on the matrix component of T.

a) If $\Delta \geq 0$ express the eccentricity of $\mathcal{E}$ in terms of a, b, c, d.

b) If $\Delta < 0$ find the angle between the asymptotes in terms of a, b, c, d.

3. The Cartesian equation $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ represents the general conic through the origin. This equation can be expressed in the matrix form

$$\begin{pmatrix} x & y \end{pmatrix}\left[Q\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D \\ E \end{pmatrix}\right] = 0$$

where $Q = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$. If $\det Q \neq 0$ the conic is either an ellipse or hyperbola (possibly degenerate). Show that the center of the conic in this case is (h, k) where

$$\begin{pmatrix} h \\ k \end{pmatrix} = -\frac{1}{2}Q^{-1}\begin{pmatrix} D \\ E \end{pmatrix}.$$

4. Let $\mathcal{E}$ be given by the Cartesian equation $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$. In each of the following cases, determine the affine type of the conic and find a transformation T such that $\mathcal{E} = \mathcal{E}(T, O)$. **Note:** There are infinitely many choices for T in each case.

a) $x^2 - xy + y^2 + x + y = 0$

b) $x^2 - 4xy + y^2 + x + y = 0$

c) $x^2 - 4xy + 4y^2 + x + y = 0$

Projective transformations: $T = [A]$

Images of Lines: $L = \langle a, b, c \rangle$

Assume $P = [\mathbf{x}] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in L$, then $L\mathbf{x} = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

Now, $P' = T(P) = [A\mathbf{x}] = [\mathbf{x}'] \in L' = T(L)$,

thus, $\left[A^{-1}\mathbf{x}'\right] \in L$, so $LA^{-1}\mathbf{x}' = 0$, that is, $\langle LA^{-1} \rangle = L'$.

Example: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $L = \langle a, b, c \rangle$

$L' = \langle LA^{-1} \rangle = \langle a, b-a, c-b \rangle$

For example, $L = \langle 2, 1, 3 \rangle$ contains $P = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $P' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in L' = \langle 2, -1, 2 \rangle$.

Definition. A *quadrilateral* is a set of four points no three of which are collinear. The set $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, U = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is called the *quadrilateral of reference*.

Fundamental Theorem of Projective Geometry (FTPG)

Theorem. *Given quadrilaterals $ABCD$ and $A'B'C'D'$, there is a unique $T \in P(2)$ such that $T : ABCD \mapsto A'B'C'D'$.*

Proof. Suffices to show $XYZU \mapsto ABCD$ for a unique collineation T_1. Then, if $T_2 : XYZU \mapsto A'B'C'D'$, $T = T_2T_1^{-1}$ is the unique collineation. First note that $\begin{pmatrix} \lambda a_1 & \mu b_1 & \nu c_1 \\ \lambda a_2 & \mu b_2 & \nu c_2 \\ \lambda a_3 & \mu b_3 & \nu c_3 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the most general matrix that sends $XYZ \mapsto ABC$.

Since

$$\begin{pmatrix} \lambda a_1 & \mu b_1 & \nu c_1 \\ \lambda a_2 & \mu b_2 & \nu c_2 \\ \lambda a_3 & \mu b_3 & \nu c_3 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda a_1 + \mu b_1 + \nu c_1 \\ \lambda a_2 + \mu b_2 + \nu c_2 \\ \lambda a_3 + \mu b_3 + \nu c_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

We have now determined λ, μ, ν up to scalar multiple:

$$\begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}^{adj} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}. \blacksquare$$

Example: If $A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ then

$$\begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $T = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ takes $XYZU$ to $ABCD$.

Factoring $T \in P(2)$ into perspective transformations

$T = [A] \in P(2)$ is a *perspective transformation* provided there are planes Π and Π' such that $A : \Pi \mapsto \Pi'$ is an isometry.

Examples

1. $T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then any Π parallel to xy-plane suffices with $\Pi' = \Pi$. In fact, if Π is any plane and Π' is its image by A then Π and Π' are isometrically related since Π' is just the rotation of Π about the z-axis through the angle θ.

2. $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$ then take any Π parallel to xy-plane and $\Pi' = \Pi + c\mathbf{k}$, a translation of Π in the direction of the z-axis.

Theorem. *Every $T \in P(2)$ is the composite of three or fewer perspective transformations.*

Proof. For any vector $\mathbf{x} \in \mathbb{R}^3$ let $\mathbf{x}_u = \frac{1}{|\mathbf{x}|}\mathbf{x}$, and for any two independent vectors $\mathbf{x}$ and $\mathbf{y}$ let $\langle \mathbf{x}, \mathbf{y} \rangle$ is the 2-space spanned by $\mathbf{x}$ and $\mathbf{y}$. Let $T = [A]$ and suppose $A : \mathbf{i}, \mathbf{j}, \mathbf{k} \mapsto \mathbf{a}, \mathbf{b}, \mathbf{c}$.

First, let $A_1 : \mathbf{i}, \mathbf{j}, \mathbf{k} \mapsto \mathbf{a}, \mathbf{b}_u, (\lambda\mathbf{b} + \mu\mathbf{c})_u$, where λ, μ are chosen so that $\mathbf{b} \cdot (\lambda\mathbf{b} + \mu\mathbf{c}) = 0$.

Note: A_1 maps the plane $x = 1$ isometrically onto its image since $A_1 (\mathbf{i} + p\mathbf{j} + q\mathbf{k}) = \mathbf{a} + p\mathbf{b}_u + q (\lambda\mathbf{b} + \mu\mathbf{c})_u$. Now, the distance between $\mathbf{i} + p_1\mathbf{j} + q_1\mathbf{k}$ and $\mathbf{i} + p_2\mathbf{j} + q_2\mathbf{k}$ is

$$|(p_1 - p_2)\,\mathbf{j} + (q_1 - q_2)\,\mathbf{k}|$$

and the distance between their images is

$$|(p_1 - p_2)\,\mathbf{b}_u + (q_1 - q_2)\,(\lambda\mathbf{b} + \mu\mathbf{c})_u|$$

These distances are equal because $\mathbf{b}_u$ and $(\lambda\mathbf{b} + \mu\mathbf{c})_u$ are orthogonal unit vectors.

Next, let $A_2 : \mathbf{a}, \mathbf{b}_u, (\lambda\mathbf{b} + \mu\mathbf{c})_u \mapsto \mathbf{a}, \mathbf{b}_u, \mathbf{c}$.

Note: A_2 maps the plane $(\lambda\mathbf{b} + \mu\mathbf{c})_u + \langle \mathbf{a}, \mathbf{b} \rangle$ isometrically onto its image $\langle \mathbf{a}, \mathbf{b} \rangle + \mathbf{c}$.

Finally, let $A_3 : \mathbf{a}, \mathbf{b}_u, \mathbf{c} \mapsto \mathbf{a}, \mathbf{b}, \mathbf{c}$.

Note: A_3 maps the plane $\langle \mathbf{a}, \mathbf{c} \rangle + \mathbf{b}_u$ isometrically onto its image $\langle \mathbf{a}, \mathbf{c} \rangle + \mathbf{b}$.

Then $A = A_3A_2A_1 : \mathbf{i}, \mathbf{j}, \mathbf{k} \mapsto \mathbf{a}, \mathbf{b}, \mathbf{c}$, and so T is the composite $T_3T_2T_1$ of the perspective transformations $T_1 = [A_1]\,, T_2 = [A_2]\,, T_3 = [A_3]$. ■

If A is an orthogonal matrix then $T = [A]$ is a perspective transformation (Example 1). However, not every perspective transformation is produced by an orthogonal matrix (Example 2). The decomposition of a collineation into perspective transformations is not unique.

Exercise. Decompose $T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ into perspective transformations.

Collinearity and Concurrence

Given two distinct points $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the line they determine is

$$L = \left\langle \begin{array}{ccc} a_2b_3 - a_3b_2 & -a_1b_3 + b_1a_3 & a_1b_2 - a_2b_1 \end{array} \right\rangle$$

A third point $C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is on this line provided $\left\langle \begin{array}{ccc} a_2b_3 - a_3b_2 & -a_1b_3 + b_1a_3 & a_1b_2 - a_2b_1 \end{array} \right\rangle \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} =$ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, that is, three points are collinear provided any three vectors that span them are linearly dependent, equivalently, the three 1-spaces belong to a common 2-space. Similarly, given two distinct lines $L_1 = \left\langle \begin{array}{ccc} a_1 & b_1 & c_1 \end{array} \right\rangle$ and $L_2 = \left\langle \begin{array}{ccc} a_2 & b_2 & c_2 \end{array} \right\rangle$, their intersection

$$L_1 \cap L_2 = \begin{bmatrix} b_1c_2 - b_2c_1 \\ a_2c_1 - a_1c_2 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

since the vector that spans this point is orthogonal to the normal vector that defines each line. A third line $L_3 = \left\langle \begin{array}{ccc} a_3 & b_3 & c_3 \end{array} \right\rangle$ contains this point provided $\left\langle \begin{array}{ccc} a_3 & b_3 & c_3 \end{array} \right\rangle \begin{bmatrix} b_1c_2 - b_2c_1 \\ a_2c_1 - a_1c_2 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, that is, the three lines are concurrent provided any three normal vectors that define them are linearly dependent. Collinearity and concurrence are the foundation of the Principle of Duality in the projective plane.

Desargues's Theorem. *Let ABC and $A'B'C'$ be triangles in $\mathbb{R}P^2$ such that AA', BB', CC' are concurrent, and let $AB \cap A'B' = R, BC \cap B'C' = P, CA \cap C'A' = Q$. Then P, Q, R are collinear.*

Proof.

First, note that if $A = A', B = B'$ or $C = C'$ then the theorem is trivial; so assume $A' \neq A, B' \neq B, C' \neq C$.

Second, note that ABC and the point of concurrence form a quadrilateral since the intersections of corresponding sides are single points. Thus, by the FTPG we can let $XYZU$ be this quadrilateral.

Now, since $A = X$ the line $AU = \langle\ 0 \quad 1 \quad -1\ \rangle$ and so $A' = \begin{bmatrix} p \\ 1 \\ 1 \end{bmatrix}$, for some choice of p, since $A' \neq A$.

Since $B = Y$ the line $BU = \langle\ 1 \quad 0 \quad -1\ \rangle$ and so $B' = \begin{bmatrix} 1 \\ q \\ 1 \end{bmatrix}$, for some choice of q, since $B' \neq B$.

Since $C = Z$ the line $CU = \langle\ 1 \quad -1 \quad 0\ \rangle$ and so $C' = \begin{bmatrix} 1 \\ 1 \\ r \end{bmatrix}$, for some choice of r, since $C' \neq C$.

Next, since $AB = XY = \langle\ 0 \quad 0 \quad 1\ \rangle, BC = YZ = \langle\ 1 \quad 0 \quad 0\ \rangle, CA = ZX = \langle\ 0 \quad 1 \quad 0\ \rangle$

and

$A'B' = \langle\ 1-q \quad 1-p \quad pq-1\ \rangle$
$B'C' = \langle\ qr-1 \quad 1-r \quad 1-q\ \rangle$
$C'A' = \langle\ 1-r \quad pr-1 \quad 1-p\ \rangle$

it follows that

$$R = \begin{bmatrix} p-1 \\ -q+1 \\ 0 \end{bmatrix}$$
$$P = \begin{bmatrix} 0 \\ -q+1 \\ r-1 \end{bmatrix}$$
$$Q = \begin{bmatrix} -p+1 \\ 0 \\ r-1 \end{bmatrix}$$

and $\begin{vmatrix} p-1 & 0 & -p+1 \\ -q+1 & -q+1 & 0 \\ 0 & r-1 & r-1 \end{vmatrix} = 0.$ ■

Corollary to the proof. *The steps in the calculation are reversible, so if P, Q, R are collinear then AA', BB', CC' are concurrent.* (This is an example of a theorem whose converse is also its dual.)

Corollary. *Let ABC be a triangle in the Euclidean plane and let L_A, L_B, L_C be cevians through the respective vertices with $L_A \cap \overleftrightarrow{BC} = A', L_B \cap \overleftrightarrow{CA} = B', L_C \cap \overleftrightarrow{AB} = C'$. Then $AB \cap A'B', BC \cap B'C', CA \cap C'A'$ are collinear if and only if L_A, L_B, L_C are concurrent.* (What is the interpretation of this corollary if, for example, $\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}$? What does the corollary say if ABC is equilateral and L_A, L_B, L_C are its medians?)

Pappus' Theorem. *Let A, B, C be distinct collinear points and let A', B', C' be distinct and collinear on a different line. Let $P = BC' \cap B'C, Q = CA' \cap C'A, R = AB' \cap A'B$. Then P, Q, R are collinear.*

Proof. By the FTPG, can assume $AA'PR = XYZU$, because, clearly neither A nor A' can be collinear with P, R, whereas if A, A', P' are collinear then neither C, C' nor B, B' could be collinear with any one of P, Q, R. Then $AR = XR = \left\langle\ 0 \quad 1 \quad -1\ \right\rangle$ whereby $B' = \begin{bmatrix} r \\ 1 \\ 1 \end{bmatrix}$ since $B' \in AR$ and $B' \neq A$.

Similarly, $A'R = YR = \left\langle\ 1 \quad 0 \quad -1\ \right\rangle$ so $B = \begin{bmatrix} 1 \\ s \\ 1 \end{bmatrix}$.

Next, $C = AB \cap B'P = XB \cap B'Z = \left\langle\ 0 \quad 1 \quad -s\ \right\rangle \cap \left\langle\ 1 \quad -r \quad 0\ \right\rangle = \begin{bmatrix} rs \\ s \\ 1 \end{bmatrix}$

and $C' = BP \cap A'B' = BZ \cap YB' = \left\langle\ s \quad -1 \quad 0\ \right\rangle \cap \left\langle\ 1 \quad 0 \quad -r\ \right\rangle = \begin{bmatrix} r \\ rs \\ 1 \end{bmatrix}$.

But then $Q = AC' \cap A'C = XC' \cap YC = \left\langle\ 0 \quad 1 \quad -rs\ \right\rangle \cap \left\langle\ 1 \quad 0 \quad -rs\ \right\rangle = \begin{bmatrix} rs \\ rs \\ 1 \end{bmatrix}$. Thus P, Q, R are collinear, in fact on $\left\langle\ 1 \quad -1 \quad 0\ \right\rangle$.■

Corollary: Brianchon's Theorem. *Let L, M, N be distinct concurrent lines and let L', M', N' be distinct and concurrent at a different point. Let $(M \cap N')(M' \cap N) = A, (N \cap L')(N' \cap L) = B, (L \cap M')(L' \cap M) = C$. Then the lines A, B, C are concurrent.*

Proof. This is the dual of Pappus' Theorem.

Note that Brianchon's Theorem is not the converse of Pappus' Theorem. It is related to Pascal's Theorem, which will be studied after our classification of conics. In that context Brianchon's Theorem can be stated as a theorem about degenerate conics: *If the six lines of a hexagon pass alternately through two distinct points then the three lines through opposite vertices are concurrent.*

Cross-Ratio and Embedding Planes

If A, B, C, D are collinear points (we will assume they are distinct) then the 1-spaces representing any two of them are spanned by linear combinations of the other two. For example, let

$$\begin{aligned} C &= [\alpha \mathbf{a} + \beta \mathbf{b}] \\ D &= [\gamma \mathbf{a} + \delta \mathbf{b}] \end{aligned}$$

where $A = [\mathbf{a}]\,, B = [\mathbf{b}]$. The ratio

$$(ABCD) = \frac{\beta\gamma}{\alpha\delta}$$

is independent of the vectors we choose to span the 1-spaces because any non-zero multiples would cancel out. This ratio is invariant if we apply any collineation $T \in P(2)$ since T can be represented by a projective linear transformation, that is, $(ABCD) = (A'B'C'D')$, where $P' = T(P)$ for any point P. We call $(ABCD)$ the *cross-ratio* of the ordered set of points A, B, C, D. Since there are $4! = 24$ permutations of the four points the cross-ratio could have 24 different values for four given collinear points. The group of permutations is generated by transpositions and it is easy to show that if $\chi = (ABCD)$ then

$$\begin{aligned} (BACD) &= (ABDC) = \frac{1}{\chi} \\ (ACBD) &= (DBCA) = 1 - \chi \end{aligned}$$

It follows that each value is stable under at least four of the permutations, so there are at most six different values. Therefore, for most purposes we consider the cross-ratio to be an equivalence class of values

$$[\chi] = \left\{ \chi, \frac{1}{\chi}, 1 - \chi, \frac{1}{1-\chi}, 1 - \frac{1}{\chi}, \frac{\chi}{\chi - 1} \right\}$$

For example, $[3] = \left\{-2, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\right\}$, however $[-1] = \left\{2, \frac{1}{2}, -1\right\}$. The second example is unique for having only three members in the equivalence class. Points that produce this cross-ratio are said to form a *harmonic set*. Cross-ratio was discovered from the theory of perspective, in terms of the following theorem that shows how to construct a collineation from local information.

Theorem. *If two sets of four points, collinear on distinct lines, are in perspective from a point then they have the same cross-ratio.*

Proof. Let A, B, C, D be distinct collinear points and let A', B', C', D' be distinct and collinear on a different line, with AA', BB', CC', DD' concurrent at E. We will show that this configuration determines a collineation T. Then, since cross-ratio is a projective invariant it will follow that $(ABCD) = (A'B'C'D')$. First, note that $BCB'C'$ and $B'C'BC$ are quadrilaterals and so, by the FTPG, there is a unique collineation T such that $T : BCB'C' \mapsto B'C'BC$. We now show that $T(A) = A'$ and $T(D) = D'$. Note also that T is an involution ($T = T^{-1}$) because the composition of T with itself leaves the points B, C, B', C' fixed and so this composition is the identity transformation. In particular, T interchanges the lines BC and $B'C'$ and so their point of intersection is fixed. Call this point F. Also, the lines BB' and CC' are invariant under T so the point E is also fixed. (*Remark.* This type of argument is commonly used in geometry. In one case a point of intersection is fixed because the lines themselves are invariant, in the other case because the lines are interchanged.) Now $T(A) = X$ is on $B'C'$. Let $G = AX \cap BB'$ and $H = AX \cap CC'$. Then G and H are fixed by T , because T is an involution so AX is also an invariant line. We now know that E, F, G, H are all fixed points, so if they are a quadrilateral T would be the identity transformation, which it is not. But E, F, G, H are a quadrilateral unless $X = A'$. Thus $T(A) = A'$ and similarly $T(D) = D'$. ■

The assignment $A \mapsto A', B \mapsto B', C \mapsto C', D \mapsto D'$ is called a *perspectivity.* We say that the two sets of points are in perspective from the point E.

Corollary to the proof. *Any perspectivity can be extended uniquely to a collineation.*

Note also that cross-ratio locates points on a line in relation to three given points on the line, because $(ABCX) = (ABCY)$ if and only if $X = Y$. As a consequence, if A, B, C, D and A, E, F, G are two sets of collinear points on different lines and $(ABCD) = (AEFG)$ then BE, CF, DG are concurrent. This observation provides:

Another Proof of Pappus' Theorem. Let V be the intersection of the two given lines. Let $D = BA' \cap AC'$ and $E = BC' \cap CA'$. Then V, A', B', C' are in perspective with B, A', R, D from A, and in perspective with B, E, P, C' from C. Thus

$$(VA'B'C') = (BEPC') = (BA'RD)$$

and so $A'E, RP, DC'$ are concurrent. Thus RP contains Q since $A'E = A'C$ and $DC' = AC'$. ■

The idea of cross-ratio was originally used to assign affine coordinates to points on a line. This is related to the usual parameterization of a line in terms of vectors. If we write

$$\begin{aligned} \mathbf{c} &= \lambda \mathbf{a} + (1-\lambda)\,\mathbf{b} \\ \mathbf{d} &= \mu \mathbf{a} + (1-\mu)\,\mathbf{b} \end{aligned}$$

then $\frac{1-\lambda}{\lambda}$ is the ratio $\frac{AC}{CB}$ and $\frac{1-\mu}{\mu}$ is the ratio $\frac{AD}{DB}$, and these ratios were interpreted as the affine coordinates of C and D relative to the points A and B with which they are collinear. Further,

$$\begin{aligned} (ABCD) &= \frac{\mu\,(1-\lambda)}{\lambda\,(1-\mu)} \\ &= \frac{AC}{CB}\frac{DB}{AD} \end{aligned}$$

Thus, the cross-ratio and its permuted values can be calculated simply from a consistent assignment of a unit of distance on the given line. Note that the ratios are oriented and so must be interpreted consistently as signed values. The affine coordinate 0 indicates that the point to be located is one of the given points, for example, $\lambda = 1$ implies $C = A$. The connection with projective geometry arises when we note that, for example, $\lambda = 0$ implies $C = B$. The affine coordinate is undefined in this case, whereby it is convenient to assign it the value ∞. More generally, assume that A, B, C, D are points on an affine line obtained as images in an embedding plane of points on a line of $\mathbb{R}P^2$. If the projective line were the 2-space parallel in $\mathbb{R}^3$ to the embedding plane then there would be no affine line containing these points. Otherwise, one of the points on the projective line would be the ideal point for the corresponding affine line. Without leaving the affine line we can still compute the cross-ratio when either A, B, C, D is this ideal point. The computation is consistent with our idea of limits for rational functions:

$$\begin{aligned} (ABCD) &= \frac{AC}{CB}\frac{DB}{AD} \\ &= \frac{DB}{CB},\ A \text{ ideal} \\ &= \frac{AC}{AD},\ B \text{ ideal} \\ &= \frac{DB}{DA},\ C \text{ ideal} \\ &= \frac{CA}{CB},\ D \text{ ideal} \end{aligned}$$

Exercise. Let $\mathcal{E}$ be an affine hyperbola or an ellipse that is not a circle. On the line through the foci let C be the center of the conic, A be one of the foci, B be the vertex closer to A, and let D be the ideal point. Show that $(ABCD)$ is the eccentricity of the conic. How could cross-ratio be used to determine the eccentricity of a circle or parabola?

Important Skills for Midterm Exam

I. Points and Lines of $\mathbb{R}P^2$

Representation of projective points $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and lines $\langle \; a \quad b \quad c \; \rangle$
Find line determined by collinear points
Find point of intersection of concurrent lines

II. Projective Collineations

Find image of a point, $T(P)$, and of a line, $T(L)$, for $T \in P(2)$
Find the inverse of a projective transformation

III. Basic Theorems and Their Affine Interpretations

Fundamental Theorem of Projective Geometry:
Determine a projective transformation from a quadrilateral and its image
Desargues's Theorem:
Determine whether two triangles are in perspective; find perspective point/line
Pappus's Theorem:
Find Pappus line for two collinear triples
Duality and Brianchon's Theorem

IV. Cross-Ratio

Calculation and invariance under projective transformation
Effect of permutation

The Cross-Product Map

Given $\mathbf{p} \in \mathbb{R}^3$ consider the linear map $\mathbf{x} \mapsto \mathbf{p} \times \mathbf{x}$. If $p = (a, b, c)$ then the matrix representing this map in terms of the standard basis is

$$\widehat{P} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

This matrix is anti-symmetric, i.e., the sum of the matrix and its transpose is the zero matrix. It is occasionally useful to identify a point P of $\mathbb{R}P^2$ with a projective matrix that takes any point X to the line through P and X. Thus, if $P = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $\widehat{P} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ is this matrix. If $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is distinct from P then $X^t \widehat{P} = L_X$, the line through P that contains X. We use this fact below to define conics as figures in $\mathbb{R}P^2$.

Conics

Let P be a point of $\mathbb{R}P^2$ and let $T \in P(2)$ with $T(P) \neq P$. The *conic at P afforded by T* is

$$\mathcal{E}(T, P) = \{L \cap T(L) : P \in L\}$$

Because of the duality between points and lines in $\mathbb{R}P^2$ it is possible to find $\mathcal{E}(T, P)$ without using Cartesian equations. Since $T(L) = LT^{adj}$ it follows that $L \cap T(L)$ is found by taking the cross-product of L and LT^{adj}. For example, let $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and suppose

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

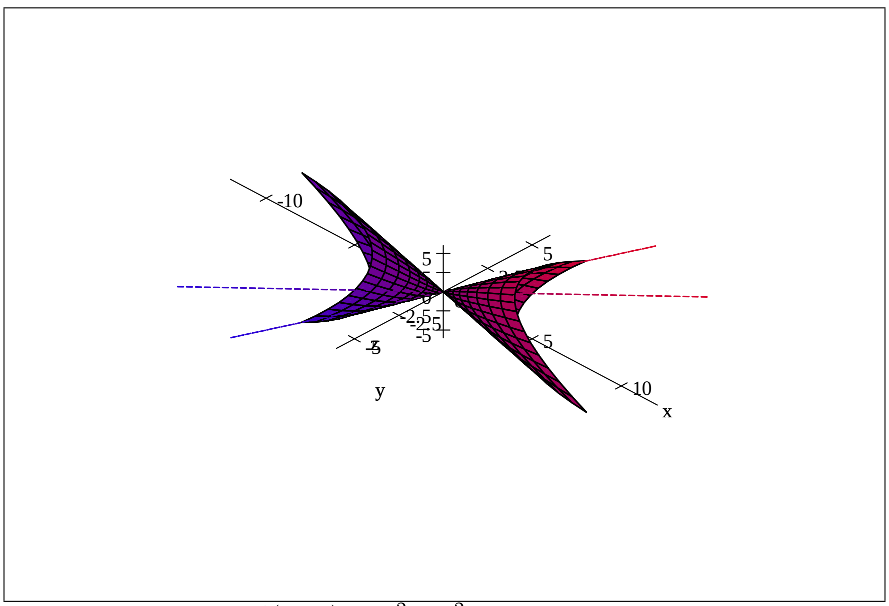
Then $T(P) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Any line through P is $L = \left\langle \; p \quad q \quad -p-q \; \right\rangle$ and so $T(L) =$

$$\begin{aligned} & \left\langle \; p \quad q \quad -p-q \; \right\rangle \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ = & \left\langle \; p \quad -p+q \quad -p-2q \; \right\rangle \end{aligned}$$

Thus $L \cap T(L) = \begin{bmatrix} pq + p^2 + q^2 \\ -pq \\ p^2 \end{bmatrix}$. The conic $\mathcal{E}(T, P)$ is this collection of points in $\mathbb{R}P^2$, e.g., the points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are seen to be on the conic by taking $p = 0$ and $q = 0$, respectively. Otherwise, dividing by pq and setting $t = \frac{p}{q}$, every point on the conic other than $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is parameterized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t + t^2 + 1 \\ -t \\ t^2 \end{bmatrix}$. Note that $t = -1$ produces P. (What value of t produces $T(P)$?) It follows that

$$y^2 + z^2 - xz - yz = 0$$

is the Cartesian equation for $\mathcal{E}(T, P)$. Finding the Cartesian equation from a parametric representation can be difficult, and so we will want to interpret the construction in a way that avoids this problem.



$\mathcal{E}(T, P)$ as $y^2 + z^2 - xz - yz = 0$

It is possible that the line L through P and $T(P)$ is invariant under T. In this case $L \cap T(L) = L$, and so the line L is contained in the conic and we say that

$\mathcal{E}(T,P)$ is *degenerate*. As an example, let T be as above but let $P = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $T(P) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $L = \langle\ 0 \quad 0 \quad 1\ \rangle$, so $T(L) = L$. It follows that $\mathcal{E}(T,P)$ is just the line L or a pair of lines $\{L, M\}$. **Exercise.** Which is it in this case?

Now consider the conic $y^2 + z^2 - xz - yz = 0$ again. In the embedding plane $z = 1$ we see the affine conic $y^2 - x - z = -1$, a parabola. Of course, in another embedding plane we may see a conic of different affine type. Recall the affine conic construction for a conic at $O = (0,0)$: Let T be the affine transformation defined by $T(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$ and let $\mathcal{E} = \{L \cap T(L) \mid O \in L\}$. Then $\mathcal{E} = \mathcal{E}(T,O)$ is the Cartesian curve $cx^2 + (d-a)xy - by^2 = (cp - aq)\,x + (dp - bq)\,y$. If we view the Cartesian plane as the embedding plane $z = 1$, the corresponding projective collineation is

$$T = \begin{bmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{bmatrix}$$

and $O = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}P^2$. Then $T(O) = \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$. The pencil through O consists of the lines $L = \langle\ s \quad t \quad 0\ \rangle$, whereby $T(L) =$

$$\begin{aligned} & \langle\ s \quad t \quad 0\ \rangle \begin{bmatrix} d & -b & bq - dp \\ -c & a & -aq + cp \\ 0 & 0 & ad - bc \end{bmatrix} \\ = \ & \langle\ -ct + ds \quad at - bs \quad t\,(-aq + cp) + s\,(bq - dp)\ \rangle \end{aligned}$$

Thus $L \cap T(L) =$

$$\begin{aligned} & \langle\ s \quad t \quad 0\ \rangle \times \langle\ -ct + ds \quad at - bs \quad t\,(-aq + cp) + s\,(bq - dp)\ \rangle \\ = & \begin{bmatrix} st\,(bq - dp) + t^2\,(-aq + cp) \\ st\,(aq - cp) + s^2\,(-bq + dp) \\ -bs^2 + ct^2 + st\,(a - d) \end{bmatrix} \end{aligned}$$

$= \begin{bmatrix} t\,(bq - dp) + t^2\,(-aq + cp) \\ -bq + dp + t\,(aq - cp) \\ -b + ct^2 + t\,(a - d) \end{bmatrix}$ if $s \neq 0$. Note that these points satisfy

$$cx^2 + (d-a)xy - by^2 - (cp - aq)\,xz - (dp - bq)\,yz = 0$$

which is just the homogeneous form of the corresponding affine curve in $z = 1$. The ideal points of this conic relative to $z = 1$ are found by setting $z = 0$ to obtain

$$cx^2 + (d - a)xy - by^2 = 0$$

Exercise. Show that there are no ideal points if $\Delta > 0$, one ideal point if $\Delta = 0$, and two ideal points if $\Delta < 0$.

Transformation of Conics

Given a conic $\mathcal{E}$ in $\mathbb{R}P^2$ and a collineation $S \in P(2)$ we would expect that $S(\mathcal{E})$ is also a conic. This follows from the general transformation principle for a figure described by a Cartesian equation. For example, let $\mathcal{E}$ be the conic $y^2 + z^2 - xz - yz = 0$ and let

$$S = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -3 & 2 \end{bmatrix}$$

If the point $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is on $S(\mathcal{E})$ then $S^{-1}(P)$ is on $\mathcal{E}$, and so

$$\begin{aligned} & \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ = & \begin{bmatrix} 3x - 5y + z \\ x + y - z \\ 3x - y + z \end{bmatrix} \end{aligned}$$

must satisfy $y^2 + z^2 - xz - yz = 0$. After substituting and simplifying we find that $S(\mathcal{E})$ is the conic

$$x^2 - 6xy + y^2 - z^2 = 0$$

Note that $S(\mathcal{E})$ cannot be degenerate unless $\mathcal{E}$ is, for otherwise S^{-1} would transform a degenerate conic into a non-degenerate conic, and this is not possible since a degenerate conic is either a single line or a pair of lines and S^{-1} is a collineation.

Symmetric Matrix Representation of Conics

There is a more direct way to obtain the transformation of a conic. Since $\mathcal{E}$ is a homogeneous quadric in $\mathbb{R}^3$ it is the zeros of a Cartesian equation

$$Ax^2 + Bxy + Cy^2 + Fxz + Gyz + Hz^2 = 0$$

which can be written in matrix form

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2A & B & F \\ B & 2C & G \\ F & G & 2H \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Thus $\mathcal{E}$ can be identified with the projective symmetric matrix $Q = \begin{bmatrix} 2A & B & F \\ B & 2C & G \\ F & G & 2H \end{bmatrix}$ and we can abbreviate its Cartesian equation as $X^tQX = 0$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Now if $S \in P(2)$ and X is a point on $\mathcal{E}$ then $S(X)$ is on $S(\mathcal{E})$. We can summarize this relation as follows

$$\begin{aligned} X^tQX &= 0 \\ S(X)^tQ'S(X) &= 0 \end{aligned}$$

where Q' is the symmetric matrix representing $S(\mathcal{E})$. To find Q' note that $S(X)$ is the matrix product SX and so $S(X)^t = X^tS^t$ which yields

$$X^t \left(S^tQ'S\right) X = 0$$

and this equation holds if and only if $S^tQ'S = Q$. It follows that

$$\begin{aligned} Q' &= \left(S^t\right)^{-1} QS^{-1} \\ &= S_{co}QS^{adj} \end{aligned}$$

For example, if $\mathcal{E}$ is the conic $y^2 + z^2 - xz - yz = 0$ and $S = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -3 & 2 \end{bmatrix}$ then $Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and $S_{co}QS^{adj} =$

$$\begin{aligned} &\begin{bmatrix} 3 & 1 & 3 \\ -5 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \\ = &\begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = Q' \end{aligned}$$

Thus Q' corresponds to the Cartesian form $x^2 - 6xy + y^2 - z^2 = 0$, in agreement with the previous calculation.

Now we can put this all together with our original definition of $\mathcal{E}$ as $\mathcal{E}(T, P)$. First we represent $\mathcal{E}(T, P)$ as a symmetric matrix Q. If X is a point other than P then $X^t\widehat{P}$ is the line L_X through X and P, so

$$\begin{aligned} &X^t\widehat{P}T^{adj} \\ = \;&L_XT^{adj} \\ = \;&T(L_X) \end{aligned}$$

If X is also on $T(L_X)$ then

$$\begin{aligned} & T(L_X)X \\ = \; & X^t \widehat{P} T^{adj} X \\ = \; & 0 \end{aligned}$$

Now $\left(X^t \widehat{P} T^{adj} X\right)^t = -X^t T_{co} \widehat{P} X$, because $\widehat{P}$ is anti-symmetric. Finally, $X^t \widehat{P} T^{adj} X + \left(X^t \widehat{P} T^{adj} X\right)^t = 0 = X^t \left(\widehat{P} T^{adj} - T_{co} \widehat{P}\right) X$, and $\widehat{P} T^{adj} - T_{co} \widehat{P}$ is symmetric. It follows that

$$Q = \widehat{P} T^{adj} - T_{co} \widehat{P}$$

Consider again $\mathcal{E}(T, P)$ where $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Here $\widehat{P} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ and so $Q =$

$$\begin{aligned} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \\ = \; & \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \end{aligned}$$

which is the conic $y^2 + z^2 - yz - xz = 0$. Note that we were able to obtain $\mathcal{E}$ without first parameterizing its points.

Finally, if $S \in P(2)$ and $\mathcal{E} = \mathcal{E}(T, P)$ note that $S(\mathcal{E}) = \mathcal{E}(STS^{-1}, S(P))$ because the pencils of lines through P and $T(P)$ are sent to the pencils of lines through $S(P)$ and $ST(P)$, respectively. Consequently, the symmetric matrix representing $S(\mathcal{E})$ is

$$\begin{aligned} & S_{co} Q S^{adj} \\ = \; & S_{co} \left(\widehat{P} T^{adj} - T_{co} \widehat{P}\right) S^{adj} \\ = \; & S_{co} \widehat{P} \left(ST\right)^{adj} - \left(ST\right)_{co} \widehat{P} S^{adj} \end{aligned}$$

MATH 529 - Second Assignment

Present a complete solution to **one** of the following:

1. Let $T \in P(2)$ be represented by the projective matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & \mu \end{bmatrix}$. Express T as the composition of three or fewer perspective transformations. Explain why each factor is a perspective transformation by exhibiting the planes in $\mathbb{R}^3$ that are related isometrically.

2. Let ABC be an affine triangle and L a line not through any vertex. If $X = L \cap AB, Y = L \cap BC, Z = L \cap CA$ then, by dropping perpendiculars to L from A, B, C and using similar triangles with signed ratios, it is easy to show

$$\frac{AX}{XB}\frac{BY}{YC}\frac{CZ}{ZA} = -1$$

(This result is often attributed to Menelaus of Alexandria, who lived about 400 years after Euclid.) Without assigning coordinates, use Menelaus's Theorem together with Desargues's Theorem to prove Ceva's Theorem: *If the cevians* L_A, L_B, L_C *of* ABC *are concurrent, with* $P = L_A \cap BC, Q = L_B \cap CA, R = L_C \cap AB$, *then*

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$$

Suggestion. Set up perspectivities from X, Y, Z and compare cross-ratios.

3. Let A, B, C be distinct points on the line L in $\mathbb{R}P^2$. Let V be a point not on L and P be a point on VC other than V or C. Let $C' = AB \cap QR$, where $Q = AP \cap VB$ and $R = BP \cap VA$. Show that C' does not depend on the choice of V and P. **Suggestion.** Show that there is no loss in generality in taking $A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then compute $(ABCC')$.

4. The points $A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $A' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are on the projective conic $\mathcal{E}$: $x^2 + y^2 - z^2 = 0$. Let L_0 be the tangent to $\mathcal{E}$ through A and let L_1 be the line through A and A'. For any line L through A other than L_0 or L_1, let P be the other intersection point of L with $\mathcal{E}$. Let L' be the line through P and A'. For all L through A, find $T \in P(2)$ such that $T(L) = L'$. Then find $T(L_0)$ and $T(L_1)$. **Suggestion.** Look at how the slope of L' is related to the slope of L in the embedding plane $z = 1$.

5. Let $\mathcal{E}$ be a conic and let X be a point outside $\mathcal{E}$. Let the tangents to $\mathcal{E}$ through X meet $\mathcal{E}$ at the points E and F. For any point P on $\mathcal{E}$ other than E or F, let $Q = XP \cap EF$ and let P' be the other point of $\mathcal{E}$ on XP. Show that the Points X, Q, P, P' form a harmonic set. **Suggestion.** To simplify the computation, first show that there is no loss of generality in taking $\mathcal{E}$ to be the conic $x^2 + y^2 - z^2 = 0$ and $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Congruence of Projective Conics

Theorem. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be projective conics of the same degeneracy type (both non-degenerate, both a single line, or both a pair of lines). Then there exists $T \in P(2)$ such that $T(\mathcal{E}_1) = T(\mathcal{E}_2)$.*

Before providing the proof we note that the theorem is obvious if both conics are degenerate, since there are infinitely many projective collineations that take one given line to another, or even a pair of given lines to another pair. We provide two proofs of the theorem, one based on perspective transformations, the other using symmetric matrices.

Proof 1. Assume $\mathcal{E}_1$and $\mathcal{E}_2$ are non-degenerate and let their intersections with the standard ($z = 1$) embedding plane Π be E_1 and E_2, respectively. Let $\mathcal{C}$ be a projective conic such that the plane curve E_1 is represented as $A_1(\Pi) \cap \mathcal{C}$ and the plane curve E_2 is represented as $A_2(\Pi) \cap \mathcal{C}$, where A_1 maps Π to $A_1(\Pi)$ isometrically and A_2 maps Π to $A_2(\Pi)$ isometrically. (It is easy to show that such a $\mathcal{C}$ exists; see B/E/G 4.1.4 for details.) Let T_1 and T_2 be the perspective transformations associated with A_1 and A_2, respectively. Then $T_1(\mathcal{E}_1) = \mathcal{C}$ and $T_2(\mathcal{E}_2) = \mathcal{C}$, so $T_2^{-1} \circ T_1(\mathcal{E}_1) = \mathcal{E}_2$. ■

Proof 2. Let Q_1 and Q_2 be the symmetric matrices representing $\mathcal{E}_1$and $\mathcal{E}_2$, respectively. Assuming again that $\mathcal{E}_1$and $\mathcal{E}_2$ are non-degenerate we have $\det Q_1 \neq 0$ and $\det Q_2 \neq 0$. Thus, the eigenvalues λ_j, μ_j, ν_j ($j = 1, 2$) of these matrices are non-zero real numbers. The matrix version of the real spectral theorem from linear algebra states that a matrix Q is symmetric if and only if there is an orthogonal matrix M such that

$$M^{-1}QM$$

is diagonal. An orthogonal matrix has the property that $M^{-1} = M^t$, so we can let $M = S^{adj}$ for some collineation S, and then $M^{-1}QM = M^tQM = S_{co}QS^{adj} =$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

where λ, μ, ν are the eigenvalues of Q, all non-zero if $\det Q \neq 0$. If Q represents the conic $\mathcal{E}$ then $S(\mathcal{E})$ has

$$Q' = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

as its matrix, that is, any non-degenerate conic is projectively congruent to a conic whose matrix is diagonal with non-zero diagonal entries. The conic

$$\lambda x^2 + \mu y^2 + \nu z^2 = 0$$

is empty in $\mathbb{R}P^2$ if λ, μ, ν all have the same sign. Assume, then, that two of these are positive and one is negative, say $\lambda = a^2, \mu = b^2, \nu = -c^2$, and let

$$D = \begin{bmatrix} \sqrt{\frac{a}{bc}} & 0 & 0 \\ 0 & \sqrt{\frac{b}{ac}} & 0 \\ 0 & 0 & \sqrt{\frac{c}{ab}} \end{bmatrix}$$

assuming a, b, c all positive. Then

$$D_{co} Q' D^{adj} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It follows that any non-degenerate, non-empty conic is projectively congruent to $x^2 + y^2 - z^2 = 0$, and therefore $T(\mathcal{E}_1) = \mathcal{E}_2$ for some $T \in P(2)$. ■

Degenerate Conics and Empty Conics

The conic $x^2 + y^2 + z^2 = 0$ is empty in $\mathbb{R}P^2$ and so cannot be congruent to the standard conic $x^2 + y^2 - z^2 = 0$. The above proof demonstrates why this is the case. If they were congruent there would be $S \in P(2)$ such that

$$S_{co} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{adj} = S_{co} S^{adj} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

However, S_{co} and S^{adj} are transposes of each other so this would imply that the dot product of a column in the matrix with itself is negative, and that cannot happen over the real numbers. These two conics would be congruent in $\mathbb{C}P^2$, the usual domain for the theory of algebraic curves, so we do not want to call this empty conic degenerate. What then is the test for when a conic is degenerate?

Theorem. *Let $\mathcal{E}$ be a projective conic with symmetric matrix Q. Then $\mathcal{E}$ is degenerate if and only if* $\det Q = 0$.

Proof. If $\mathcal{E}$ is non-degenerate then $\det Q \neq 0$ because there exists $S \in P(2)$ such that $Q = S_{co} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} S^{adj}$, which has non-zero determinant. Conversely, if $\mathcal{E}$ is degenerate then its Cartesian form is a product of linear factors

$$(ax + by + cz)(px + qy + rz) = 0$$

and so

$$Q = \begin{pmatrix} 2ap & aq+bp & ar+cp \\ aq+bp & 2bq & br+cq \\ ar+cp & br+cq & 2cr \end{pmatrix}$$

and it is straightforward to show that $\det Q = 0$.

Exercise. Show that $x^2 + y^2 + z^2$ is not the product of linear factors, even if complex coefficients are allowed.

Determination of Conics from Given Points

A basic result from the theory of non-degenerate algebraic curves states that the number of points in common with any given line is at most the degree of the curve. If the curve $\mathcal{E}$ is a conic we can see this directly. We have seen that if $\mathcal{E}(T, P)$ is non-degenerate then it acquires a parameterization from the matrix calculation to find its points. If the parameter is t, then with the possible exception of a single point the coordinates are given by quadratic polynomials in t. When these coordinates are substituted into the linear equation for a given line the result is a quadratic equation in t, which of course has at most two real solutions. For example, since any two non-degenerate conics are projectively congruent, assume that $\mathcal{E}$ is $xy + yz + zx = 0$. If $L = \left\langle \begin{array}{ccc} a & b & c \end{array} \right\rangle$ then $L \cap \mathcal{E}$ is the set of points $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$\begin{aligned} ax + by + cz &= 0 \\ xy + yz + zx &= 0 \end{aligned}$$

One parameterization (see p. 184 of B/E/G) of $\mathcal{E}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t^2 + t \\ t + 1 \\ -t \end{bmatrix}$$

which yields every point on $\mathcal{E}$ except $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so the intersections of L with $\mathcal{E}$ are obtained from the values of t such that

$$a\left(t^2 + t\right) + b\,(t+1) - ct = 0$$

Specifically, $t = \frac{1}{2a}\left(c - a - b \pm \sqrt{a^2 + b^2 + c^2 - 2\left(ab + 2ac + bc\right)}\right)$, if $a \neq 0$, which has $0, 1$ or 2 values depending on the discriminant. Finally, note that $a = 0$ if and only if $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is on L, in which case $b\left(t+1\right) - ct = 0$ has a unique solution if $b \neq c$ and no solution otherwise.

Corollary. *Any non-degenerate conic is determined by five distinct points.*

Proof. Let $\mathcal{E}$ be a non-degenerate conic. Since any line intersects $\mathcal{E}$ in at most two points any five distinct points of $\mathcal{E}$ will have the property that no three are collinear. Using the FTPG and the fact that any two non-degenerate conics are projectively congruent, let four of the given points be the quadrilateral of reference $XYZU$ and let $P = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ be the fifth point. Then p, q, r are non-zero and distinct (otherwise P would be on a line of the quadrilateral). As we deduced earlier, $\mathcal{E}$ has Cartesian equation

$$\begin{aligned} Bxy + Fxz + Gyz &= 0 \\ B + F + G &= 0 \end{aligned}$$

and since $P \in \mathcal{E}$ we also have

$$Bpq + Fpr + Gqr = 0$$

from which we conclude that $\mathcal{E}$ must be the conic

$$r\left(p - q\right)xy + q\left(r - p\right)xz + p\left(q - r\right)yz = 0$$

Note, then, that there are infinitely many conics through any four given points, no three of which are collinear. Also, it follows that there is a unique affine conic through any five given points, no three of which are collinear.

Another theorem should be established at this juncture. We know that any two non-degenerate conics are projectively congruent but that the collineation transforming one into another is not unique. Given the nondegenerate conics $\mathcal{E}_1$ and $\mathcal{E}_2$, how specific can we be in prescribing a mapping from $\mathcal{E}_1$ to $\mathcal{E}_2$? As we have seen, $\mathcal{E}_1$ and $\mathcal{E}_2$ may both contain $XYZU$ yet be distinct conics, so specifying the images of four points, no three of which are collinear, may not produce a collineation that maps $\mathcal{E}_1$ to $\mathcal{E}_2$. What is the maximum number of points on $\mathcal{E}_1$ whose images we can prescribe on $\mathcal{E}_2$?

Theorem. *Given P_1, Q_1, R_1 on $\mathcal{E}_1$ and P_2, Q_2, R_2 on $\mathcal{E}_2$ there exists $T \in P(2)$ with $T(\mathcal{E}_1) = \mathcal{E}_2$ such that $T : P_1Q_1R_1 \mapsto P_2Q_2R_2$.*

Proof. Let $T_1 \in P(2)$ with $T_1 : P_1Q_1R_1 \mapsto XYZ$. Then $T_1(\mathcal{E}_1)$ is

$$Bxy + Fxz + Gyz = 0$$

and B, F, G are non-zero because $T_1(\mathcal{E}_1)$ must be non-degenerate. Now let

$$T_2 = \begin{bmatrix} \frac{1}{G} & 0 & 0 \\ 0 & \frac{1}{F} & 0 \\ 0 & 0 & \frac{1}{B} \end{bmatrix}$$

Then $T_2 \circ T_1 : P_1Q_1R_1 \mapsto XYZ$ and $T_2 \circ T_1(\mathcal{E}_1)$ is $xy + xz + yz = 0$.Similarly,

there are collineations T_3 and T_4 such that $T_4 \circ T_3 : P_2Q_2R_2 \mapsto XYZ$ and $T_4 \circ T_3(\mathcal{E}_2)$ is also $xy + xz + yz = 0$.

It follows that $T_3^{-1} \circ T_4^{-1} \circ T_2 \circ T_1 = T$. ■

Polarity

Let $\mathcal{E}$ be a non-degenerate conic represented as the level surface $s(x,y,z)=0$, where $s(x,y,z)=Ax^2+Bxy+Cy^2+Fxz+Gyz+Hz^2$. Then $\nabla s\,(x,y,z)= Q\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $Q=\begin{pmatrix} 2A & B & F \\ B & 2C & G \\ F & G & 2H \end{pmatrix}$. Since the gradient of a scalar function is perpendicular to its level sets at the point of evaluation it follows that the tangent line L to $\mathcal{E}$ at $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is the 2-space with normal vector $Q\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, so

$$L=\left\langle\ 2Ax_1+By_1+Fz_1 \quad Bx_1+2Cy_1+Gz_1 \quad Fx_1+Gy_1+2Hz_1\ \right\rangle$$

For example, if $s(x,y,z)=xy+xz+yz$ then the tangent line to $\mathcal{E}$ at $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is $L=\langle\ 1 \quad 1 \quad 4\ \rangle$, the 2-space $x+y+4z=0$.

The process of associating a point on $\mathcal{E}$ with a line determined by $\mathcal{E}$ can be generalized to any point in $\mathbb{R}P^2$, resulting in a map $P\mapsto L_P$, and this induces a dual map $L\mapsto P_L$. The conic $\mathcal{E}$ determines this map that interchanges points and lines, which is called the *polarity* induced by $\mathcal{E}$. We call L_P the *polar* of P and P_L the *pole* of L. A polarity is characterized by the following property: If $P\in L$ then $P_L\in L_P$. We say that the polarity *reverses inclusion.* It also follows that a polarity induced by $\mathcal{E}$ is an involution.

Theorem. *Let $\mathcal{E}$ be a non-degenerate conic represented by the symmetric matrix Q. For any point P of $\mathbb{R}P^2$ let $L_P=P^tQ$ and for any line L of $\mathbb{R}P^2$ let $P_L=Q^{adj}L^t$. Then the correspondence $P\mapsto L_P, L\mapsto P_L$ is a polarity.*

Proof. We must show that if L is a line through P then P_L is on L_P, that is, $L_PP_L=0$ if $LP=0$. Now $L_PP_L=(P^tQ)\left(Q^{adj}L^t\right)=P^t\left(QQ^{adj}\right)L^t=P^tL^t$, since QQ^{adj} is non-zero because $\det Q\neq 0$. However, $P^tL^t=(LP)^t=0$.

This is the polarity induced by the non-degenerate conic $\mathcal{E}$. What is it's geometric interpretation? First, if $P\in\mathcal{E}$ then L_P is the tangent L to $\mathcal{E}$ at P, and if L is a tangent to $\mathcal{E}$ the P_L is its point of contact. The conic $\mathcal{E}$ segregates the remaining points of $\mathbb{R}P^2$ into those *outside* and those *inside.*

Definition. We say P is *outside* $\mathcal{E}$ if there exists a line L through P that is exterior to $\mathcal{E}$, that is, such that $L\cap\mathcal{E}=\varnothing$. We say P is *inside* $\mathcal{E}$ if every line through P intersects $\mathcal{E}$ in two distinct points, that is, every L through P is a secant.

Exercise. Let P be a point not on $\mathcal{E}$. If P is outside $\mathcal{E}$ choose a representation of Q such that $P^tQP > 0$. Then $E^tQE > 0$ for any point E that is outside $\mathcal{E}$, and $E^tQE < 0$ for any point E that is inside $\mathcal{E}$.

Consider the case where $P = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is outside $\mathcal{E}$, and let E be a point different from P. Then $E^t\widehat{P}$ is the line through E and P and, if this line is tangent to $\mathcal{E}$ at E, then $E^t\widehat{P} = E^tQ$. Conversely, if $E^t\widehat{P} = E^tQ$ then $E^t\widehat{P}E = E^tQE$, and $E^t\widehat{P}E = 0$ since E is on the line $E^t\widehat{P}$, so E is on $\mathcal{E}$. Suppose, then, that we want to find a point E on $\mathcal{E}$ such that the line through P is tangent to $\mathcal{E}$ at E, that is, we want to find a point E such that $E^t\widehat{P} = E^tQ$. As a vector equation in $\mathbb{R}^3$ this implies $E^t\left(\widehat{P} + \lambda Q\right) = \mathbf{0}$ for some $\lambda \neq 0$. Thus, we can find E by determining the values of λ such that $\det\left(\widehat{P} + \lambda Q\right) = 0$ and then finding the kernel of $\widehat{P} + \lambda Q$. There will be two such values of λ, corresponding to the two points E_1 and E_2 on $\mathcal{E}$ such that the lines $E_1^t\widehat{P}$ and $E_2^t\widehat{P}$ are tangents to $\mathcal{E}$. The line $E_1^t\widehat{E}_2$, through E_1 and E_2, will be L_P, so if P is outside $\mathcal{E}$ then its polar is a secant for $\mathcal{E}$. Conversely, if L is a secant then P is outside $\mathcal{E}$, being the intersection of the two tangents at $L \cap \mathcal{E}$. For example, If $\mathcal{E}$ is the conic $x^2 + y^2 - z^2 = 0$ and $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ then $\widehat{P} + \lambda Q = \begin{bmatrix} \lambda & -1 & 1 \\ 1 & \lambda & -1 \\ -1 & 1 & -\lambda \end{bmatrix}$, whose determinant is $\lambda(1-\lambda)(\lambda+1)$, whereby the values $\lambda = \pm 1$ yield the points $E_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ such that $E_1^t\left(\widehat{P} - Q\right) = 0$ and $E_2^t\left(\widehat{P} + Q\right) = 0$, respectively. Then $E_1^t\widehat{P} = \langle\ 1 \quad 0 \quad -1\ \rangle$ and $E_2^t\widehat{P} = \langle\ 0 \quad 1 \quad -1\ \rangle$ are the tangents to $\mathcal{E}$ through P, and L_P is the secant $\langle\ 1 \quad 1 \quad -1\ \rangle$.

This construction for L_P is equivalent to the definition in the above theorem, but in the process it also produced the pair of tangents through P and their points of contact. Historically it was this construction, in the affine plane, that motivated the idea of polarity and its projective completion. This affine construction is associated with the nineteenth century geometer Joachimsthal due to his introduction of an efficient notation for describing the intersections of lines with algebraic curves. Let $s = 0$ be an affine conic given by

$$s = Ax^2 + Bxy + Cy^2 + +Fx + Gy + H$$

For a line L through two given points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, Joachimsthal wanted to describe systematically the relation of L to the conic $s = 0$. A typical parameterization of L is given by

$$\begin{aligned} x &= (1-t)\,x_2 + tx_1 \\ y &= (1-t)\,y_2 + ty_1 \end{aligned}$$

and by letting $t = \frac{1}{k+1}$, where k locates a point P on L according to the ratio $k = \frac{P_1P}{PP_2}$, the intersections of L with the conic are obtained from the roots of a relatively simple quadratic

$$s_{22}k^2 + 2s_{12}k + s_{11} = 0$$

The s_{ij} are called the Joachimsthal coefficients, systematically expressed as functions of P_1, P_2 and the coefficients that determine the conic (see p. 167 of B/E/G and p. 175 for their projective versions). **Exercise:** *Find the coefficients s_{ij} in terms of the coordinates of P_1, P_2.* Thus L is a secant if there are two distinct values of k that solve this equation, a tangent if one value with multiplicity 2, and an exterior line if no real value of k solves the equation. The Joachimsthal coefficients were precursors of tensor notation in that the format of the subscripts determines the type of expression. Since a double subscript produces a scalar quantity and no subscript, s, produces a quadratic in x and y, it was logical to use a single subscript to produce a linear expression in x and y (again, see p. 167). Joachimsthal noticed that, if P_i is on the conic, the equation $s_i = 0$ is the equation of the tangent line through P_i (note that its projective version is equivalent to our determination of L_P in this case), and he used this observation to generalize the interpretation for any point P_i in the plane. For example, if P_1 is a point through which it is possible to construct a tangent to the conic then in fact there will be two such tangents through P_1. If P_2 is any other point on such a tangent line then the Joachimsthal discriminant

$$s_{12}^2 - s_{22}s_{11}$$

must be zero, and since the only restriction on P_2 is that it be on this line the corresponding discriminant must be identically zero as P_2 varies. Apparently this is indicated in this notation by dropping the subscript 2 wherever it appears so that a quadratic expression in x, y results. The restriction on P_2 requires that this expression

$$s_1^2 - s \cdot s_{11}$$

be a product of distinct linear factors. Setting each factor equal to zero provides the two tangents to the conic through P_1. This method is equivalent to our determination of the lines $E_1^t\widehat{P}$ and $E_2^t\widehat{P}$ in the projective setting above. It also follows that the line through the two points of contact from P_1 in Joachimsthal's method corresponds to the polar line L_P when P is outside the projective conic. Thus the polar of P_1 is the secant line $s_1 = 0$ because the points of contact satisfy $s = 0$.

The same Joachimsthal equation $s_1 = 0$ should also produce the polar of P_1 in the third case, when it is not possible to construct a tangent to the conic through

P_1. In this case the discriminant $s_{12}^2 - s_{22}s_{11}$ is positive for any point $P_2 \neq P_1$, indicating two points of intersection with the conic. We will demonstrate this projectively. Let P be inside $\mathcal{E}$ and suppose a given line through P intersects $\mathcal{E}$ at E_1 and E_2. Then the polar of $E_1$is the tangent line $E_1^t Q$ and the polar of E_2 is the tangent line $E_2^t Q$. By taking the cross-product we obtain the intersection of these tangent lines. The following result from linear algebra, which is straightforward to verify, is useful.

Theorem. *Let X be an ordered triple and let Q* be a symmetric matrix*. Then* $\left[\widehat{QX}\right] = \left[Q^{adj}\widehat{X}Q^{adj}\right]$.

As usual, the square brackets indicate projective matrices. Since $\left(E_j^t Q\right)^t = QE_j$ we can now express the intersection of the two tangents as

$$\begin{aligned} & \widehat{QE}_1 QE_2 \\ = \;& Q^{adj}\widehat{E}_1 Q^{adj} QE_2 \\ = \;& Q^{adj}\widehat{E}_1 E_2 \end{aligned}$$

Now $L_P = P^t Q$ and

$$\begin{aligned} & \left(P^t Q\right) Q^{adj}\widehat{E}_1 E_2 \\ = \;& P^t \left(\widehat{E}_1 E_2\right) \\ = \;& 0 \end{aligned}$$

because $\left(\widehat{E}_1 E_2\right)^t$ is the line through $E_1$and E_2 and this line was assumed to pass through P. It follows that the polar of P is the line exterior to $\mathcal{E}$ consisting of the intersections of all pairs of tangents whose points of contact are the intersections of $\mathcal{E}$ with lines through P, and that the pole of any secant line to $\mathcal{E}$ is the intersection of the tangents at its two points in common with $\mathcal{E}$. Further, if L is a line exterior to $\mathcal{E}$ then the polars of all of its points are concurrent at a point inside $\mathcal{E}$.

This geometric interpretation of polarity induced by a non-degenerate conic is usually referred to as **La Hire's Theorem**. Polarity often suggests proofs to theorems about conics that do not require reduction to coordinates:

Theorem. *Let A, B, C, D be points on a non-degenerate conic $\mathcal{E}$, and suppose that the tangents to $\mathcal{E}$ at A and C meet at P on BD. Show that the tangents to $\mathcal{E}$ at B and D meet on AC.*

Proof. Since $P = L_A \cap L_C = P_{AC}$ is on BD it follows that $L_B \cap L_D = P_{BD}$ is on AC. ■

Theorem. *Let A, B, C be points on a non-degenerate conic $\mathcal{E}$, and let $L_A \cap BC = E, L_B \cap CA = F, L_C \cap AB = G$.* Then E, F, G are collinear.

Proof. We obtain the dual of this theorem by polarity:

Let L, M, N be tangents to $\mathcal{E}$ with $P_L = A, P_M = B, P_N = C$. Then the lines $P_{BC}A, P_{CA}B, P_{AB}C$ are concurrent.

This dual theorem is easily proved by transforming $\mathcal{E}$ to a conic whose affine representation in an embedding plane is a circle. Then $P_{BC}A, P_{CA}B, P_{AB}C$ are cevians of the triangle $P_{BC}P_{CA}P_{AB}$ and the following segments are equal

$$\begin{aligned} P_{AB}A &= BP_{AB} \\ P_{BC}B &= CP_{BC} \\ P_{CA}C &= AP_{CA} \end{aligned}$$

Then

$$\frac{P_{AB}A}{AP_{CA}} \cdot \frac{P_{CA}C}{CP_{BC}} \cdot \frac{P_{BC}B}{BP_{AB}} = 1$$

and so $P_{BC}A, P_{CA}B, P_{AB}C$ are concurrent by Ceva's Theorem. The theorem now follows by duality. ■

Diagonal Triangle of a Quadrilateral

Definition. The *diagonal triangle* of a quadrilateral $ABCD$ is EFG, where $E = AB \cap CD, F = AC \cap BD, G = AD \cap BC$.

Theorem. *Let the non-degenerate conic $\mathcal{E}$ contain the vertices of a quadrilateral $ABCD$. Then the tangents to $\mathcal{E}$ at any pair of points of the quadrilateral intersect on a line of its diagonal triangle.*

Proof. Since $E = AB \cap CD$ its polar L_E contains the pole of AB and the pole of CD. However, the pole of AB is $L_A \cap L_B$, the intersection of the tangents at A and B, and the pole of CD is $L_C \cap L_D$, the intersection of the tangents at C and D. Similarly, L_G is the line through $L_B \cap L_C$ and $L_A \cap L_D$, and L_F is the line through $L_A \cap L_C$ and $L_B \cap L_D$. We can transform $\mathcal{E}$ to the conic $xy + yz + zx = 0$ with $ABC \mapsto XYZ$. Then D is mapped to $\left[t^2 + t, t + 1, -t\right]$ for some $t \notin \{-1, 0\}$, and it is readily seen that $L_E = FG, L_F = GE, L_G = EF$. ■

Dual Construction of a Conic

The concept of polarity allows us to define a conic as its collection of tangents. This collection is usually called a *line conic*. We obtain line conics by dualizing our original construction of a conic at the point P afforded by the collineation T.

Definition. Let L be a line in $\mathbb{R}P^2$ and let $T \in P(2)$ such that $T(L) \neq L$. The collection of lines $\mathcal{E}(T,L) = \{PT(P) : P \in L\}$ is *the line conic at* L *afforded by* T.

This line conic should be represented by a symmetric matrix Q' such that if X is a line of $\mathcal{E}(T,L)$ then $XQ'X^t = 0$. If Q is the symmetric matrix representing the conic consisting of the poles P_X of the lines X then

$$P_X^t Q P_X = 0$$

However, $P_X^t = XQ^{adj}$ so

$$\begin{aligned} & P_X^t Q P_X \\ = & \left(XQ^{adj}\right) Q \left(Q^{adj} X^t\right) \\ = & XQ^{adj} X^t \end{aligned}$$

It follows that $Q' = Q^{adj}$. To find Q' from T and L, suppose that the line X is the line through P and $T(P)$, where $P \in L$. Then $P = \widehat{L}X^t$ and $T(P) = T\widehat{L}X^t$ and so

$$\begin{aligned} X\widehat{L}X^t &= 0 \\ &= XT\widehat{L}X^t \end{aligned}$$

Therefor, $X\left(\widehat{L} - T\widehat{L}\right)X^t = 0$ and $\left(X\left(\widehat{L} - T\widehat{L}\right)X^t\right)^t = X\left(\widehat{L}T^t - \widehat{L}\right)X^t$. Consequently,

$$\begin{aligned} Q' &= \left(\widehat{L} - T\widehat{L}\right) + \left(\widehat{L}T^t - \widehat{L}\right) \\ &= \widehat{L}T^t - T\widehat{L} \end{aligned}$$

and the symmetric matrix representing the conic consisting of the poles of this line conic is the adjoint of Q'. As an example, let

$$\begin{aligned} T &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ L &= \left\langle \begin{array}{ccc} 1 & 0 & 0 \end{array} \right\rangle \end{aligned}$$

The points $P = \begin{bmatrix} 0 \\ p \\ q \end{bmatrix}$ are the points of L and $T(P) = \begin{bmatrix} p+q \\ p+q \\ q \end{bmatrix}$ and so $\mathcal{E}(T,L)$ consists of the lines $X = \left\langle \begin{array}{ccc} q^2 & -q\,(p+q) & p\,(p+q) \end{array} \right\rangle$. Since $\widehat{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ we have $Q' = \widehat{L}T^t - T\widehat{L} =$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

from which we can verify that $XQ'X^t = 0$. The adjoint of Q' is

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

and so $\mathcal{E}(T, L)$ consists of the tangents to the conic

$$y^2 + z^2 - 4xz + 2yz = 0$$

In particular, L is the tangent at the point $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Pascal's Theorem

Let A, B, C, A', B', C' be six distinct points on a non-degenerate conic $\mathcal{E}$ in $\mathbb{R}P^2$ and let $AB' \cap BA' = P, BC' \cap CB' = Q, CA' \cap AC' = R$. Then P, Q, R are collinear.

Proof. Since no three of the points are collinear we can assume $\mathcal{E}$ is the conic $xy + yz + zx = 0$ and that $A = X, B = Y, C = Z$. Then $A' = \begin{bmatrix} p+p^2 \\ p+1 \\ -p \end{bmatrix}, B' = \begin{bmatrix} q+q^2 \\ q+1 \\ -q \end{bmatrix}, C' = \begin{bmatrix} r+r^2 \\ r+1 \\ -r \end{bmatrix}$ for distinct p, q, r not equal to 0 or -1, and so $P = \begin{bmatrix} q(p+1) \\ q+1 \\ -q \end{bmatrix}, Q = \begin{bmatrix} q(r+1) \\ r+1 \\ -q \end{bmatrix}, R = \begin{bmatrix} p(r+1) \\ r+1 \\ -r \end{bmatrix}$. Since

$$\det \begin{bmatrix} q(p+1) & q(r+1) & p(r+1) \\ q+1 & r+1 & r+1 \\ -q & -q & -r \end{bmatrix} = 0$$

it follows that P, Q, R are collinear. ■

Pascal's Theorem is a generalization of Pappus's Theorem since two distinct lines in $\mathbb{R}P^2$ constitute a degenerate conic. It is the strongest possible generalization in the following sense.

Theorem. *Let A, B, C, A', B', C' be six distinct points in $\mathbb{R}P^2$, no three of which are collinear, and let $AB' \cap BA' = P, BC' \cap CB' = Q, CA' \cap AC' = R$. If P, Q, R are collinear then the six points are on a non-degenerate conic.*

Proof. By the FTPG we can assume $ABCA' = XYZU$. Let $B' = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, C' = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$, where necessarily p, q, r are distinct and non-zero, as are s, t, u. Then $P = \begin{bmatrix} r \\ q \\ r \end{bmatrix}, Q = \begin{bmatrix} ps \\ qs \\ pu \end{bmatrix}, R = \begin{bmatrix} t \\ t \\ u \end{bmatrix}$. Since X, Y, Z, U are on any non-degenerate conic $Bxy + Fxz + Gyz = 0$ with $B + F + G = 0$, we must show that B' and C' are on a conic of this form. Let $B = u(t-s), F = t(s-u), G = s(u-t)$. Then $Bxy + Fxz + Gyz = 0$ is non-degenerate and contains C'. Since P, Q, R are collinear we have

$$\begin{aligned} \det \begin{bmatrix} r & ps & t \\ q & qs & t \\ r & pu & u \end{bmatrix} &= \\ pqtu + prst + qrsu - pqsu - qrst - prtu &= 0 \end{aligned}$$

and so B' also satisfies $Bxy + Fxz + Gyz = 0$. ■

Corollary to the proof. Since any five of the six points determine a unique conic it follows that $r(q-p)xy + q(p-r)xz + p(r-q)yz = 0$ is the same conic, that is, $\begin{bmatrix} r(-p+q) \\ q(p-r) \\ p(-q+r) \end{bmatrix} = \begin{bmatrix} u(-s+t) \\ t(s-u) \\ s(-t+u) \end{bmatrix}$. Equivalently, $(u(t-s), t(s-u), s(u-t)) \times (r(q-p), q(p-r), p(r-q)) = (0,0,0)$, which itself is equivalent to $pqtu + prst + qrsu - pqsu - qrst - prtu = 0$.

Brianchon's Theorem

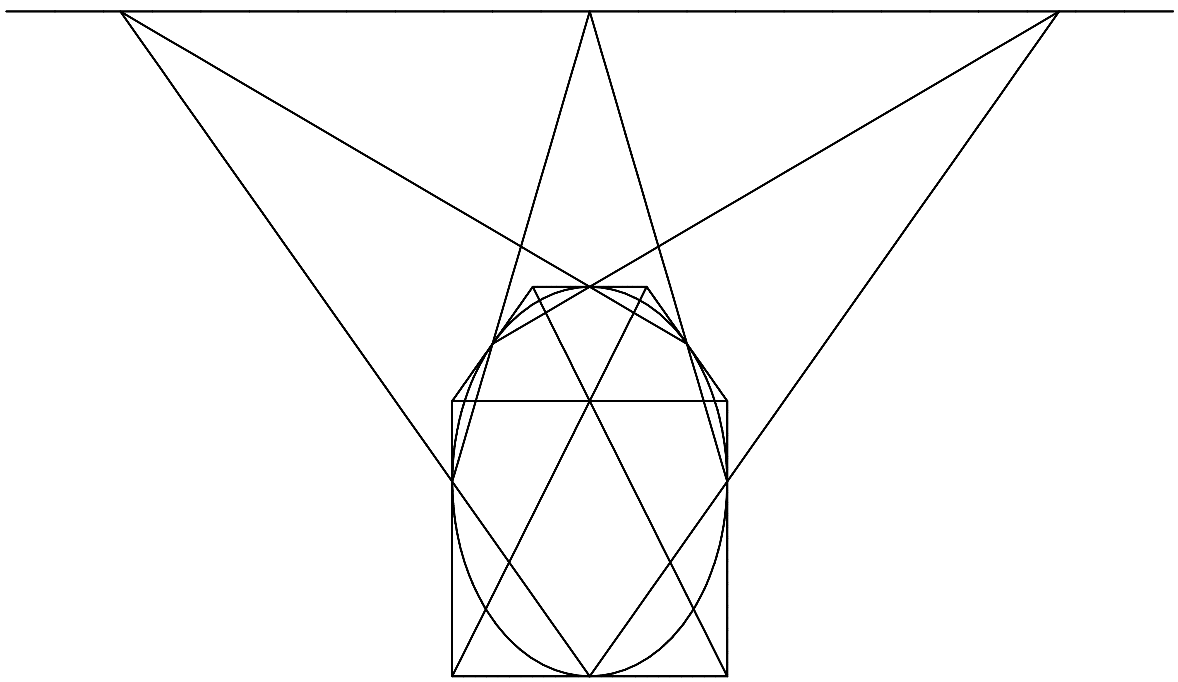
The dual of Pascal's Theorem is:

Let L, M, N, L', M', N' be six distinct tangents to a non-degenerate conic $\mathcal{E}$ in $\mathbb{R}P^2$ and let $(L \cap M')(M \cap L') = L_1, (M \cap N') \cap (N \cap M') = L_2, (N \cap L') \cap (L \cap N') = L_3$. Then L_1, L_2, L_3 are concurrent.

Charles J. Brianchon (1785-1864) originally stated this result as an affine theorem, where the duality with Pascal's Theorem expresses a basic fact about hexagons:

The three diagonals of a hexagon circumscribed about a conic are concurrent, and the three intersections of the opposite sides of a hexagon inscribed in a conic are collinear.

A possible reason for the late discovery of the dual of Pascal's Theorem is that the two results can be difficult to represent in the same diagram. The following figure shows the inscribed hexagon constructed from the points of contact of Brianchon's tangents. It follows that the three collinear points are the poles of the three diagonals and therefor the tangents from these points are found from the intersections of the diagonals with the conic.



The dual theorems of Pascal and Brianchon

Important Skills for Final Exam

I. Cross-Ratio

II. Diagonal Triangle of a Quadrilateral

III. Determination of $\mathcal{E}(T, P)$
Parametric form
Cartesian form
Symmetric matrix Q
Degenerate conics

IV. Transformation of Conics
Given T and $\mathcal{E}$, determine $T(\mathcal{E})$
Given $\mathcal{E}_1$ and $\mathcal{E}_2$, find T such that $T(\mathcal{E}_1) = \mathcal{E}_2$; specifying images of three points

V. Determination of conic from five given points

VI. Polarity
Given $\mathcal{E}$, find L_P for P and P_L for L
Given P outside $\mathcal{E}$, find tangents through P and their points of contact

VII. Theorems of Pascal and Brianchon