MATH 329-01 (82628): Transformation Geometry
JB-387, MW 6:00 - 7:50PM
SYLLABUS Fall 2011

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Text: Coxeter/Greitzer, *Geometry Revisited*
MAA New Mathematical Library
(ISBN: 0883856190)
Prerequisites: MATH 251

This course explores Euclidean geometry within the context of modern geometry where the concept of congruence is based on transformations of the plane. There are two objectives that we will pursue concurrently, important results in Euclidean geometry from the modern era and the classification of transformations of the Euclidean plane that are generated by reflections in lines. For the first objective we will cover selected topics from the first three chapters of the text in seminar style, that is, you will be presenting most of the material. For the second objective I will develop the language of transformation geometry that will allow us to classify the congruence transformations of the Euclidean plane. The midterm and final exams will be based on the transformation geometry, while the seminar work will account for the remainder of your course grade, as follows:

1) **Presentation (10%)**
2) **Write-up of presentation (20%)**
3) **Midterm exam (30%)**
4) **Final exam (40%)**

For the two in-class exams you should bring your notes since the purpose is to learn how to apply your own work to the material. Full credit for the presentation will be given provided you make an honest effort on the assigned problem. The write-up can be perfected later with a final draft due no more than two weeks after your presentation. This draft must be in typed form and can be submitted electronically.

Course grades will be determined, according to the above weighting, as follows:

	$A :\geq 91$	$A- : 86 - 90$
$B+ : 81 - 85$	$B : 76 - 80$	$B- : 71 - 75$
$C+ : 66 - 70$	$C : 61 - 65$	$C- : 51 - 60$
$D : 45 - 50$	$F :< 45$	

Some important dates:

September 26 - First day of class
September 28 - Last day to add classes over MyCoyote for Fall quarter
October 12 - Fall CENSUS; last day to submit add/drop slips
October 24 - Winter Advising begins
October 26 - Midterm exam
October 31 - Winter Priority Registration begins
November 30 - Last day of class
December 7 - Final Exam

Notes:
1) If you anticipate enrolling in **MATH 599** then be sure to prepare your presentation write-up for your portfolio. Try to get any paperwork for these entries to me as soon as possible after completion if they require my signature.
2) If you are in need of an accommodation for a disability in order to participate in this class, please contact Services to Students with Disabilities at UH-183, (909)537-5238.
3) Excluding very special circumstances, the college will not approve petitions to drop the course after **October 12**. Please refer to the Academic Regulations and Policies section of your current bulletin for information regarding add/drop procedures and consequences of academic dishonesty.

References: The following list is a small sample of texts that may be useful at this level. You are encouraged search out others. In particular, the Dover paperbacks are very affordable.

The Thirteen Books of Euclid's Elements (3 vol., Dover).
Geometry, Brannan/Esplen/Gray (Cambridge).
Introduction to Geometry, Coxeter (Wiley).
A Survey of Geometry, Eves (Allyn & Bacon).
Advanced Euclidean Geometry, Johnson (Dover).
Elementary Geometry from an Advanced Standpoint, Moise (Addison Wesley)
Geometry: A Comprehensive Course, Pedoe (Dover).
Modern Geometries, Smart (Brooks/Cole).
A Gateway to Modern Geometry (2nd edition), Stahl (Jones and Bartlett)

Some Theorems from Euclid's Elements:

The first 28 propositions from Book I do not require Postulate 5.

I.1 Construct equilateral triangle on a given segment.

I.2 Place at a given point a segment equal to a given segment.

I.3 Given two unequal segments, cut off from the greater a segment equal to the lesser.

I.4 SAS

I.5 Base angles of isosceles are equal (and exterior extensions).

I.6 Sides opposite equal base angles are equal.

I.7 Uniqueness of triangle with given base and sides in same half-plane relative to base.

I.8 SSS

I.9 Bisect an angle.

I.10 Bisect a segment.

I.11 Construct a $\perp$ at a given point on a line.

I.12 Drop a $\perp$ to a line from a given point.

I.13 Angles at intersection of two lines are supplementary.

I.14 Rays making supplementary angles at the vertex of a common ray between them are collinear.

I.15 Vertical angles are equal.

I.16 An exterior angle of a triangle is greater than either of the interior and opposite angles.

I.17 The sum of two interior angles in a triangle is less than two right angles.

I.18 In a triangle, the greater side subtends the greater angle.

I.19 Converse of **I.18**.

I.20 Strict triangle inequality.

I.21 If D is an interior point of $\triangle ABC$ then $BD + DC < BA + AC$ and $\angle BDC > \angle BAC$.

I.22 Construct a triangle from three segments of given lengths satisfying **I.20**.

I.23 Construct an angle equal to a given angle at a given point on a given line.

I.24 If in $\triangle ABC$ and $\triangle DEF$ we have $AB = DE$ and $AC = DF$ and $\angle BAC > \angle EDF$, then $BC > EF$.

I.25 Converse of I.24 (if $BC > EF$ then $\angle BAC > \angle EDF$).

I.26 ASA

I.27 Equal alternate angles implies parallel.

I.28 Equal opposite interior/exterior angles, or supplementary opposite interior angles, implies parallel.

I.29 Converses of **I.27** and **I.28** [*existence of at least one parallel through a given point*].
I.30 Lines parallel to a given line are parallel to each other.
I.31 Construct a parallel to a given line through a given point.
I.32 Exterior angle of a triangle is equal to sum of opposite interior angles and sum of interior angles is two right angles.
I.33 If $AB = CD$ and $\overline{AB} \parallel \overline{CD}$ then $AC = BD$ and $\overline{AC} \parallel \overline{BD}$.
I.34 Opposite side and angles of a parallelogram are equal and a diagonal divides the area in half.
I.35 Parallelograms with a common base and opposite side on the same line have equal areas.
I.36 Parallelograms with equal and collinear bases and opposite side on the same line have equal areas.
I.37 Triangles with a common base and third vertex on the same line have equal areas.
I.38 Triangles with equal and collinear bases and third vertex on the same line have equal areas.
I.39 If triangles ABC and DBC have equal areas then $\overline{AD} \parallel \overline{BC}$.
I.40 If triangles ABC and CDE have equal areas with $BC = CE$ and B, C, E collinear, then $\overline{AD} \parallel \overline{BC}$.
I.41 If a parallelogram and triangle have the same base and the third vertex of the triangle is on the line through the opposite side of the parallelogram, then the area of the parallelogram is twice that of the triangle.
I.42 Construct a parallelogram with a given angle equal in area to a given triangle.
I.43 Given a parallelogram and a point on a diagonal, the two parallelograms determined by this point and the other two vertices have equal areas.
I.44 Construct a parallelogram with a given angle on a given segment as side with area equal to that of a given triangle.
I.45 Construct a parallelogram with a given angle equal in area to a given rectilineal figure.
I.46 Construct a square on a given segment.
I.47 Pythagorean Theorem.
I.48 Converse of Pythagorean Theorem.

II.12 Law of Cosines (obtuse angle).
II.13 Law of Cosines (acute angle).

III.1 Construct the center of a given circle.
III.3 A diameter bisects a chord iff it is perpendicular to it.
III.4 Two chords that are diameters cannot bisect each other.
III.10 Two distinct circles have at most two points in common.
III.11 The line through the centers of tangent circles contains the tangent point (internal case).

III.12 The line through the centers of tangent circles contains the tangent point (external case).
III.13 A circle cannot be tangent to another circle at more than one point.
III.14 Equal chords are equally distant from the center, and conversely.
III.16 A perpendicular to a diameter at an endpoint falls outside the circle.
III.17 Construct a tangent to a circle from a given point.
III.18 A tangent is perpendicular to the diameter at the tangent point.
III.19 Converse of **III.18**.
III.20 A central angle with a given base arc is double a circumference angle with the same base arc.
III.21 Corollary of **III.20** (equality of circumference angles).
III.22 Opposite angles of a quadrilateral inscribed in a circle sum to two right angles.
III.25 Construct the full circle from a given arc.
III.27 Generalization of **III.21** to equal circles.
III.31 Thales's Theorem and related angles.
III.32 The supplementary angles made by a tangent at the endpoint of a chord are respectively equal to the circumference angles on the opposite sides of the chord.
III.33 Given $\overline{AB}$, construct a circle that circumscribes a triangle ABC, where the angle at C has also been given.
III.34 Given a circle, inscribe a triangle having a given angle.
III.35 If chords $\overline{AC}$ and $\overline{BD}$ intersect at E then $AE \cdot EC = BE \cdot ED$.
III.36 If $\overline{DB}$ is tangent to a circle at B and $\overline{DA}$ cuts the circle at A and C then $DA \cdot DC = DB \cdot DB$.
III.37 Converse of **III.36** (equality implies $\overline{DB}$ is a tangent).

IV.4 Inscribe a circle in a given triangle.
IV.5 Circumscribe a circle about a given triangle.
IV.6 Inscribe a square in a given circle.
IV.7 Circumscribe a square about a given circle.
IV.8 Inscribe a circle in a given square.
IV.9 Circumscribe a circle about a given square.
IV.10 Construct an isosceles triangle with each base angle twice the vertex angle.
IV.11 Inscribe a regular pentagon in a given circle.
IV.12 Circumscribe a regular pentagon about a given circle.
IV.13 Inscribe a circle in a regular pentagon.
IV.14 Circumscribe a circle about a regular pentagon.
IV.15 Inscribe a regular hexagon in a given circle.
IV.16 Inscribe a regular 15-gon in a given circle.

VI.1 Triangles and parallelograms with equal heights have areas in proportion to their bases.
VI.2 A segment between two sides of triangle is parallel to the third side iff it divides the sides proportionally.

VI.3 In a triangle, an cevian divides the opposite side in the same ratio as the sides on the vertex iff it bisects the angle at that vertex.

VI.4 Equiangular triangles are similar in that corresponding sides are proportional.

VI.5 Converse of **VI.4**.

VI.8 The altitude from a right angle creates two triangles similar to the whole.

VI.13 Construct the mean proportional of two lengths.

VI.19 For similar triangles, the ratio of areas is the square of the ratio of corresponding sides.

VI.24 Parallelograms constructed across the diagonal of a parallelogram are similar to the whole.

VI.30 Divide a segment so that the whole is to the longer as the longer is to the shorter.

VI.31 Generalized Pythagorean Theorem for right triangles.

VI.33 Basis of radian measure.

Greek Alphabet

$\boldsymbol{\alpha}, \boldsymbol{A}$	alpha	$\boldsymbol{\nu}, \boldsymbol{N}$	nu
$\boldsymbol{\beta}, \boldsymbol{B}$	beta	$\boldsymbol{\xi}, \boldsymbol{\Xi}$	ksi
$\boldsymbol{\gamma}, \boldsymbol{\Gamma}$	gamma	$\boldsymbol{o}, \boldsymbol{O}$	omicron
$\boldsymbol{\delta}, \boldsymbol{\Delta}$	delta	$\boldsymbol{\pi}, \boldsymbol{\Pi}$	pi
$\boldsymbol{\varepsilon}, \boldsymbol{E}$	epsilon	$\boldsymbol{\rho}, \boldsymbol{P}$	rho
$\boldsymbol{\zeta}, \boldsymbol{Z}$	zeta	$\boldsymbol{\sigma}, \boldsymbol{\Sigma}$	sigma
$\boldsymbol{\eta}, \boldsymbol{H}$	eta	$\boldsymbol{\tau}, \boldsymbol{T}$	tau
$\boldsymbol{\theta}, \boldsymbol{\Theta}$	theta	$\boldsymbol{\upsilon}, \boldsymbol{\Upsilon}$	upsilon
$\boldsymbol{\iota}, \boldsymbol{I}$	iota	$\boldsymbol{\varphi}, \boldsymbol{\Phi}$	phi
$\boldsymbol{\kappa}, \boldsymbol{K}$	kappa	$\boldsymbol{\chi}, \boldsymbol{X}$	chi
$\boldsymbol{\lambda}, \boldsymbol{\Lambda}$	lambda	$\boldsymbol{\psi}, \boldsymbol{\Psi}$	psi
$\boldsymbol{\mu}, \boldsymbol{M}$	mu	$\boldsymbol{\omega}, \boldsymbol{\Omega}$	omega

Available Presentation Exercises:

2.3.1 (page 35)

2.3.2 (page 36)

Proof of Theorem 2.12 (page 29)

Proof of **Morley's Theorem** (page 47)

Find a triangle such that
a) $R = 2r$
b) $R = \left(1+\sqrt{2}\right) r$
c) $R > \left(1+\sqrt{2}\right) r$
d) $2r < R < \left(1+\sqrt{2}\right) r$

Pappus's Proofs of Euclid I.5 and I.6

I.5 *In an isosceles triangle the angles at the base are equal to one another.*

Proof. Let ABC be a triangle with $AB = AC$. Then triangle ABC and triangle ACB have two sides and their respective included angles equal since $AB = AC$ and $AC = AB$ by hypothesis and $\angle BAC = \angle CAB$. Therefore, by Euclid I.4 (SAS) the remaining corresponding parts of the two triangles are equal, in particular $\angle ABC = \angle ACB$. ■

Pappus's proof demonstrated an idea that influenced much of his work with geometry, that congruence is equivalence under rigid motion. It took many centuries for this idea to be formalized and accepted. Roughly 1500 years after Pappus geometers would present his proof to **I.5** in terms of transformation geometry:

Proof. Let ABC be a triangle with $AB = AC$. Let D be the intersection of line BC with λ, the bisector of $\angle BAC$. Consider the reflection in λ. In particular, A and D are fixed and $ABC \mapsto AB'C'$, so $\angle DAB \mapsto \angle DAB'$. Since reflections preserve the magnitude of angles it follows that $\angle DAB' = \angle DAB = \angle DAC$, so B' is on the ray AC. However, $AB' = AB = AC$ because reflections preserve lengths, and so $B' = C$ since B' is on AC. Similarly, $C' = B$. Consequently $ABC \mapsto ACB$ under reflection in λ, whereby $\angle ABC \mapsto \angle ACB$ and so these angles have the same measure. ■

Pappus was also aware of the duality between measure of angles and length of segments implicit in Euclid's work (for example, **II.12** and **II.13**). The following proof of **I.6**, the converse of **I.5**, is the formalization of the one Pappus offered. It is similar in strategy to the proof of **I.5**.

I.6 *If in a triangle two angles are equal to one another, the sides opposite these angles are also equal to one another.*

Proof. Let ABC be a triangle with $\angle ABC = \angle ACB$. Consider the reflection in μ, the perpendicular bisector of segment BC. Let M be the midpoint of BC. The line BC is invariant under reflection in μ and M is fixed, so $MB = MB'$. Thus $ABC \mapsto A'CB$ since $B' = C$ and $C' = B$. Since reflections preserve measure of angles we have $\angle A'BC = \angle ACB = \angle ABC$. But then A' is on the line AB. Similarly, $\angle A'CB = \angle ABC = \angle ACB$, so A' is also on the line AC. Thus $A' = A$, whereby $AB \mapsto AC$ and so these segments have the same length. ■

A few comments on notation. The history of geometry is filled with attempts to denote common figures in a way that is consistent with assertions concerning them. Many of these systems of notation were inconsistent with one another due to changing viewpoints regarding the meaning of geometric theorems. We will adopt the approach that emphasizes a minimum of notation. For example, you will notice in the above proofs that we have occasionally conflated the separate concepts of angle as a figure with angle as a measure, so when we say $\angle ABC = \angle ACB$ we are really asserting their equivalence in terms of measurement. Mathematicians call this practice "abuse of notation" and justify it by context. Interestingly, Euclid was among the first to do this since his concept of congruence was implicitly one of equivalence. A high school textbook might distinguish between an angle and its measure by using $\angle ABC$ for the figure and $\measuredangle ABC$ for its measure, whereby we would have written $\measuredangle ABC = \measuredangle ACB$ in the hypothesis of **I.6**. However, in our transformation approach it will be clearer to let these distinctions sort themselves out by context. Instead, we will be careful about other aspects of notation, particularly those that indicate images of points and the like. Occasionally it will be helpful to use $\overline{AB}$ for the segment between A and B, AB for the length of this segment, and $\overleftrightarrow{AB}$ for the line through these two points.

Equivalents of Euclid's Fifth Postulate

There are many propositions in Euclidean plane geometry that are equivalent to **E.5**, Euclid's fifth postulate: *If the interior angles on one side of a transversal to two given lines sum to less than two right angles then the two lines intersect on that side of the transversal.* The equivalence of **E.5** to the axiom of Proclus/Playfair/Hilbert (**PPH**) and to the result that the sum of interior angles in any triangle is two right angles ($\boldsymbol{\Delta}_\pi$) is shown below. We say that two lines are *parallel* ($l \parallel m$) if there is no point in the plane that lies on both lines. (Euclid uses this word in his list of "common notions" but not in his postulates.) We will make use of Euclid's propositions **I.5**, **I.23** and **I.27**, which he proves without using **E.5**. (The converse of **I.27** is contained in **I.29**, which is the first of Euclid's propositions that requires **E.5.**) The final step also requires *Pasch's Axiom*, which in one form states that a line through a vertex of a triangle, properly within the interior angle at that vertex, must intersect the segment opposite the vertex.

E.5 $\Longrightarrow$ PPH

Assuming **E.5** we prove there is a unique line through a given point X not on a given line $\overleftrightarrow{CD}$ that is parallel to $\overleftrightarrow{CD}$. Let Y be any point on $\overleftrightarrow{CD}$ and let $\overleftrightarrow{AB}$ be a line through X such that $\angle DYX = \angle AXY$. Then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ by **I.27**. If $\overleftrightarrow{A'B'}$ is any line through X distinct from $\overleftrightarrow{AB}$, with A' in the same half-plane (determined by $\overleftrightarrow{XY}$) as A and B' in the same half-plane as B, then one of the following must be true:

$$\begin{aligned} \angle A'XY &< \angle AXY \\ \angle B'XY &< \angle BXY \end{aligned}$$

In the first case, since $\angle DYX = \pi - \angle CYX$ it follows that $\angle A'XY + \angle CYX < \pi$. In the second case, since $\angle CYX = \pi - \angle DYX$ it follows that $\angle B'XY + \angle DYX < \pi$. In either case, **E.5** implies that $\overleftrightarrow{A'B'}$ must intersect $\overleftrightarrow{CD}$. Thus $\overleftrightarrow{AB}$ is the unique line through X that is parallel to $\overleftrightarrow{CD}$.

PPH $\Longrightarrow$ $\boldsymbol{\Delta}_\pi$

Given triangle ABC, let $\overleftrightarrow{DE}$ be the unique line through A parallel to $\overleftrightarrow{BC}$, with D and E chosen such that $\angle BAD$ is alternate to $\angle ABC$ and $\angle EAC$ is alternate to $\angle BCA$. Then $\angle BAD = \angle ABC$ because any choice of point D' in the same half-plane as D (determined by $\overleftrightarrow{AB}$) such that $\angle BAD' = \angle ABC$ (which exists by **I.23**) implies $\overleftrightarrow{AD'} \parallel \overleftrightarrow{BC}$ and so $\overleftrightarrow{AD'} = \overleftrightarrow{DE}$. Similarly, $\angle EAC = \angle BCA$. It follows that $\angle ABC + \angle BCA = \angle BAD + \angle EAC$. Since $\angle BAD + \angle EAC + \angle CAB = \pi$ (straight angle) it follows that the sum of interior angles of ABC also equals π.

$\boldsymbol{\Delta}_\pi$ $\Longrightarrow$ E.5

This last step requires Mathematical Induction, which helps explain why the independence of **E.5** was disputed. Let l be a transversal that intersects $\overleftrightarrow{AB}$ at A and $\overleftrightarrow{CD}$ at C. Let $\angle ACD = \alpha$ and $\angle CAB = \beta$ and assume that $\alpha + \beta < \pi$. We will inductively define a sequence of points $\{D_n\}$: Let $D_1 = D$, let D_2 be the point such that D_1 is on $\overline{CD}_2$ with $AD_1 = D_1D_2$, and in general let D_{n+1} be the point such that D_n is on $\overline{CD}_{n+1}$ with $AD_n = D_nD_{n+1}$. By construction, AD_nD_{n+1} is isosceles with vertex D_n. Let $\delta_n = \angle AD_nC$. Then, for $n > 1$, we have $\delta_{n-1} = \pi - \angle AD_{n-1}D_n$ and, by **I.5**, $\delta_n = \angle D_nAD_{n-1}$. By hypothesis, the sum of interior angles of any triangle equals π, so $\delta_{n-1} = \pi - (\pi - 2\delta_n) = 2\delta_n$. By induction we have

$$\delta_n = \frac{1}{2^{n-1}}\delta_1$$

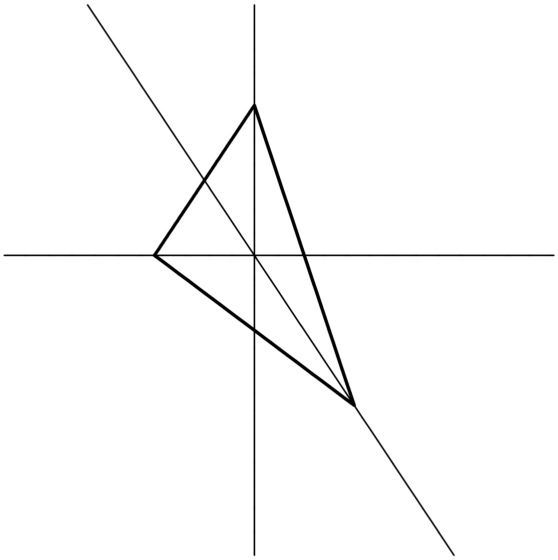
Now let $\beta_n = \angle CAD_n$. Again, by hypothesis, $\beta_n = \pi - \alpha - \delta_n$ and so $\lim_{n\to\infty} \beta_n = \pi - \alpha$ since $\lim_{n\to\infty} \delta_n = 0$. But $\pi - \alpha > \beta$ by construction, so for some large enough value of n we must have $\beta_n > \beta$. Then $\overleftrightarrow{AB}$ is properly within $\angle CAD_n$ and so, by Pasch's Axiom, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in the half-plane of l where α and β are measured.

Ceva's Theorem (1678)

Let ABC be any triangle and let L_A, L_B, L_C be three lines each of which contains only the indicated vertex of the triangle (such lines are called *cevians*). Ceva's Theorem provides a necessary and sufficient condition for these lines to be concurrent in terms of the ratios in which they divide the opposite sides of the triangle. Let $P = L_A \cap \overleftrightarrow{BC}, Q = L_B \cap \overleftrightarrow{AC}, R = L_C \cap \overleftrightarrow{AB}$, and suppose these cevians are concurrent at X. For any triangle with vertices U, V, W let (UVW) denote its area. Consider triangles ABP and APC. We have

$$\frac{BP}{PC} = \frac{(PAB)}{(PCA)}$$

because both triangles have the same height when we consider both bases to be on $\overleftrightarrow{BC}$.



For the same reason we also have

$$\frac{BP}{PC} = \frac{(PXB)}{(PCX)}$$

and so

$$\frac{BP}{PC} = \frac{(PAB) - (PXB)}{(PCA) - (PCX)} = \frac{(XAB)}{(XCA)}.$$

Similarly,

$$\begin{aligned} \frac{AR}{RB} &= \frac{(XCA)}{(XBC)} \\ \frac{CQ}{QA} &= \frac{(XBC)}{(XAB)} \end{aligned}$$

and so we conclude that

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$

Conversely, If $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1$ then the cevians must be concurrent because if X is the intersection of any two of them, for example if $L_A \cap L_B = X$, then $\overleftrightarrow{CX} \cap \overleftrightarrow{AB} = R'$ and so

$$\frac{AR}{RB} = \frac{AR'}{R'B}$$

But the ratio of distances to two given points uniquely determines the location of any point on the line through the given points. Thus, $R = R'$. ■

Note that by considering the ratios to be signed the notation is consistent with signed areas, so Ceva's Theorem is completely general in that it applies to cevians concurrent at a point exterior to the triangle as well.

Stewart's Theorem

In Figure 1.2C (page 6), let $x = \cos\angle AXB$. By the Law of Cosines,

$$\begin{aligned} c^2 &= m^2 + p^2 - 2mpx \\ b^2 &= n^2 + p^2 + 2npx \end{aligned}$$

Solving both equations for x and equating

$$\begin{aligned} b^2m + c^2n &= (m+n)\left(mn + p^2\right) \\ p^2 &= \frac{b^2m + c^2n}{a} - mn \end{aligned}$$

since $m + n = a$.

Now suppose the cevian of length q divides b into $s + t$ and the cevian of length r divides c into $h + k$. From Stewart's Theorem we have

$$\begin{aligned} a^2t + c^2s &= b\left(st + q^2\right) \\ b^2k + a^2h &= c\left(hk + r^2\right) \end{aligned}$$

If, in addition,

$$\frac{h}{k}\frac{m}{n}\frac{s}{t} = 1$$

then the three cevians are concurrent, and it follows after some algebra that

$$a^2b^2c^2 = \left(b\left(s + \frac{q^2}{t}\right) - a^2\right)\left(a\left(m + \frac{p^2}{n}\right) - c^2\right)\left(c\left(h + \frac{r^2}{k}\right) - b^2\right)$$

equivalently,

$$p^2q^2r^2 = -\left(ha + \frac{c-h}{a}\left(b^2 - hc\right)\right)\left(mb + \frac{a-m}{b}\left(c^2 - ma\right)\right)\left(sc + \frac{b-s}{c}\left(a^2 - sb\right)\right)$$

so either one or all three of the factors on the right is negative.

If the cevians are the **medians** this reduces to

$$8a^2b^2c^2 = \left(4p^2 - 2c^2 + a^2\right)\left(4q^2 - 2a^2 + b^2\right)\left(4r^2 - 2b^2 + c^2\right)$$

equivalently,

$$pqr = \frac{1}{8}\sqrt{\left(2a^2 - c^2 + 2b^2\right)\left(2b^2 - a^2 + 2c^2\right)\left(2c^2 - b^2 + 2a^2\right)}$$

In Figure 1.3D (page 9) suppose p is the length of the **angle bisector** at A. Let $BL = m$ and $LC = n$. Then

$$\begin{aligned} \frac{m}{n} &= \frac{c}{b} \\ m + n &= a \end{aligned}$$

Solving for m and n we obtain

$$\begin{aligned} m &= \frac{ac}{b+c} \\ n &= \frac{ab}{b+c} \end{aligned}$$

so by Stewart's Theorem

$$p^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]$$

Thus, if the angle bisectors have lengths p, q, r it follows that

$$pqr = \frac{abc}{(a+b)(b+c)(c+a)}\sqrt{(a+b+c)^3(a+b-c)(b+c-a)(c+a-b)}$$

Circumcenter O, Centroid G, Orthocenter H, Incenter I

Theorem 1.31 *Medians divide ABC into six triangles of equal area.*

$$\begin{aligned}(GBP) &= (GPC) = x\\ (GCQ) &= (GQA) = y\\ (GAR) &= (GRB) = z\end{aligned}$$

But also

$$\begin{aligned}(CAR) &= 2y + z\\ &= (CRB) = 2x + z\end{aligned}$$

so $x = y$ and, similarly, $y = z$. ■

Theorem 1.32 *Medians trisect each other.*

$$(GAB) = 2\,(GBP)$$

and same altitude from B implies

$$AG = 2GP$$

and, similarly,

$$\begin{aligned}BG &= 2GQ\\ CG &= 2GR\end{aligned}$$

■

Theorem 1.33 *Angle bisector divides opposite side in proportion to length of adjacent side.*

$$\begin{aligned}\frac{BP}{\sin\frac{1}{2}A} &= \frac{c}{\sin P}\\ \frac{PC}{\sin\frac{1}{2}A} &= \frac{b}{\sin P}\\ \frac{BP}{PC} &= \frac{c}{b}\end{aligned}$$

■

Corollary. *Angle bisectors are concurrent.*

$$\begin{aligned}\frac{CQ}{QA} &= \frac{a}{c}\\ \frac{AR}{RB} &= \frac{b}{a}\\ \frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} &= \frac{b}{a}\frac{c}{b}\frac{a}{c} = 1\end{aligned}$$

■

Note. The locus of points equidistant from two intersecting lines is the pair of lines bisecting the angles at the point of intersection. Thus, the intersection I (*incenter*) of the internal angle bisectors of ABC is at distance r (*inradius*) from each side of the triangle. The circle with center I and radius r is the *incircle* of ABC.

Excenters I_a, I_b, I_c and Exradii r_a, r_b, r_c

Let x, y, z be the lengths of the tangents from A, B, C, respectively, to the incircle. Let $s = \frac{1}{2}(a+b+c)$, the semi-perimeter of ABC. Then

$$\begin{aligned} x &= s-a\\ y &= s-b\\ z &= s-c\end{aligned}$$

From Ceva's Theorem it follows that the cevians drawn to the points of contact to the incircle are concurrent since

$$\frac{s-a}{s-b}\frac{s-b}{s-c}\frac{s-c}{s-a} = 1$$

The point of concurrence is called the *Gergonne point* of the triangle, and it shows up in several theorems of inversive geometry.

Theorem 1.42 $(ABC) = (IBC) + (ICA) + (IAB) = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = sr$

Corollary (*1.4.2*) From 1.1.3

$$(ABC) = \frac{abc}{4R}$$

Thus $sr = \frac{abc}{4R}$ and so

$$abc = 4srR$$

Theorem 1.43 *Let* I_a *be the intersection of the external bisectors of* B *and* C*, etc. Then* I_a *is on the internal bisector of* A*, etc.*

Proof. I_cI_a consists of points equidistant from AB and BC and I_aI_b consists of points equidistant from BC and CA, so I_a is equidistant from all three lines, in particular it is on the internal bisector of A since it is not on I_bI_c; etc. ■

Note that A, B, C are the feet of the altitudes of triangle $I_aI_bI_c$. Thus, any triangle is the *orthic triangle* of the triangle whose vertices are the intersections of its external angle bisectors.

Notation.

Let r_a be radius of excircle centered at I_a, etc. For $i, j, k \in \{a, b, c\}$ with $i \neq j$, let $s^i_{j,k}$ be the distance between a vertex of the triangle ABC and a point of contact with an excircle, where i corresponds to the side opposite the vertex, j corresponds to the side containing the point of contact, and k indicates that the excircle is centered at I_k. For example, $s^a_{b,a}$ is the distance between vertex A and the point of contact with the circle centered at I_a and the line AC. Then, since the tangents from a point to a circle have equal lengths the following 18 distances are easily determined:

$$\begin{aligned} \text{If } i &= k \text{ then } s^i_{j,k} = s. \\ \text{If } i &\neq k, \text{ then } s^i_{j,k} = s - l, \text{ where } l \notin \{i, k\}. \end{aligned}$$

For example, $s^a_{b,a} = s$ whereas $s^c_{b,a} = s - b$. As a corollary of this result, note that the cevians drawn to the points of contact with the excircles tangent to the sides opposite the vertices are concurrent because

$$\frac{s-b}{s-a}\frac{s-c}{s-b}\frac{s-a}{s-c} = 1$$

Compare this result with the construction of the Gergonne point. The symbol $s^i_{j,k}$ still makes sense if $i = j$ but its value is more complicated because it corresponds to the length of a cevian. It can be evaluated using Stewart's Theorem, with m and n in each case determined from the above formulas for $s^i_{j,k}$ with $i \neq j$ and with signs attached so that $m + n \in \{a, b, c\}$:

$$\begin{aligned} \left(s^i_{i,i}\right)^2 &= s\left(\frac{(j-k)^2}{i} + (s-i)\right), \text{ with } i, j, k \text{ distinct.} \\ \text{If } i &\neq k, \text{ then } \left(s^i_{i,k}\right)^2 = (s-j)\left(\frac{(j+k)^2}{i} - (s-k)\right), \text{ where } j \notin \{i, k\}. \end{aligned}$$

The radii of the excircles (*exradii*) are denoted r_a, r_b, r_c. Using similar right triangles we obtain

$$\begin{aligned} \frac{r_a}{r} &= \frac{s}{s-a} \\ \frac{r_b}{r} &= \frac{s}{s-b} \\ \frac{r_c}{r} &= \frac{s}{s-c} \end{aligned}$$

whereby, from 1.42, $(ABC) = rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$. The above ratios also yield the "sum of resistances" formula

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

It is useful, here, to recall the formula for the area of a triangle probably discovered by Archimedes (a generation or two after Euclid) but usually attributed to Heron of Alexandria (about 300 years later),

$$(ABC) = s(s-a)(s-b)(s-c)$$

from which we conclude

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

and then

$$(ABC)^2 = rr_ar_br_c$$

Power of a Point

Referring to Euclid **III.35** and **III.36** above, we obtain:

Theorem 2.11 *If a line through a point P intersects a circle at A and A' (not necessarily distinct, the line may be tangent) and another line through P meets the circle at B and B' (not necessarily distinct) then*

$$(PA)\,(PA') = (PB)\,(PB')$$

Although Euclid considered this expression only as a product of lengths we can interpret it as a product of signed distances. The theorem tells us that the product does not depend on the line through P that is chosen. This motivates the following definition:

> Given a circle $\mathcal{C}$ and any point P, the *power* of P with respect to $\mathcal{C}$ is $(PA)\,(PA')$.

Note that the power of P is negative if P is inside $\mathcal{C}$, zero if P is on $\mathcal{C}$, and positive if P is outside $\mathcal{C}$. For example, if P is the center of $\mathcal{C}$ then its power is $-k^2$, where k is the radius of $\mathcal{C}$. In general, by choosing the line through P that contains the center of $\mathcal{C}$ we see that the power of P is

$$d^2 - k^2$$

where d is the distance from P to the center of $\mathcal{C}$. Note that if $d > k$ so that P is outside $\mathcal{C}$ then, by the Pythagorean Theorem, $d^2 - k^2$ is the square of the length of either tangent segment from P to $\mathcal{C}$.

Also, using the Law of Sines we obtain the following result (see page 29): *The power of the incenter I of a triangle with respect to the circumcircle is*

$$-2rR = -\frac{abc}{2s}$$

Thus, if $d = OI$ then $d^2 = R\,(R - 2r)$ and so, for any triangle,

$$R \geq 2r$$

It also follows that

$$\begin{aligned} d^2 - r^2 &= R^2 - 2rR - r^2 \\ &= r^2\left(\frac{R^2}{r^2} - 2\frac{R}{r} - 1\right) \end{aligned}$$

and so O is inside the incircle provided

$$2 \leq \frac{R}{r} < 1 + \sqrt{2}$$

Radical Axis

Circles of Apollonius. *Let $k > 0$. Given distinct points A and B, determine the locus of points P such that $PA = kPB$.*

Solution. Let $\mathcal{L}$ be this locus. For any point P in the plane other than A let P' be its image after applying Steiner's construction for the circle of radius 1 centered at A. Then

$$\begin{aligned} (AB)\,(AB') &= 1 \\ (AP)\,(AP') &= 1 \\ \frac{AB}{AP'} &= \frac{AP}{AB'} \end{aligned}$$

so APB is similar to $AB'P'$. Thus

$$\begin{aligned} \frac{B'P'}{PB} &= \frac{AP'}{AB} \\ B'P' &= \frac{PB}{(AB)\,(AP)} \end{aligned}$$

If $P \in \mathcal{L}$ then $AP = kPB$ so

$$B'P' = \frac{1}{kAB}$$

i.e., P' is on the circle of radius $\frac{1}{kAB}$ centered at B'. This circle interchanged with $\mathcal{L}$, so $\mathcal{L}$ is either a circle or a line. In particular, $\mathcal{L}$ is the $\perp$-bisector of segment AB if $k = 1$ and a circle whose center is on the line AB otherwise. ■

Exercise. *If $k \neq 1$ and $d = AB$ show that the radius of $\mathcal{L}$ is*

$$\frac{kd}{|k^2 - 1|}$$

and the distance from the center of this circle to whichever point (A or B) it encloses is

$$\begin{aligned} \frac{d}{k^2 - 1}, \text{ if } k &> 1 \\ \frac{dk^2}{1 - k^2}, \text{ if } k &< 1 \end{aligned}$$

For example, if $k_1 > 1$ and $k_2 < 1$, let M be the midpoint of segment AB and, if P is on the $\perp$-bisector of AB, let $\theta = \angle MAP = \angle MBP$. If $p = PA = PB$ then $d = 2p\cos\theta$ and so

$$\begin{aligned} d_1^2 &= \left(bd + b^2 + p^2\right) \\ d_2^2 &= \left(ad + a^2 + p^2\right) \\ a &= \frac{dk_2^2}{1 - k_2^2}, r_2 = \frac{k_2 d}{1 - k_2^2} \\ b &= \frac{d}{(k_1^2 - 1)}, r_1 = \frac{k_1 d}{k_1^2 - 1} \end{aligned}$$

Then

$$\begin{aligned} d_1^2 &= bd + b^2 + p^2 \\ d_2^2 &= ad + a^2 + p^2 \end{aligned}$$

and so $d_1^2 - r_1^2 = p^2 = d_2^2 - r_2^2$. It is easy to check that this last equation holds if $k_1 > 1$ and $k_2 > 1$, and also if $k_1 < 1$ and $k_2 < 1$.

Theorem. *If P is on the $\perp$-bisector of segment AB then the power of P is the same for every circle in the Apollonian family determined by A and B.*

Proof. From the above calculation, the power of P for any circle in the family is $(PA)^2 = (PB)^2$. ■

For any two circles in the Apollonian family determined by the points A and B, the $\perp$-bisector of AB is the locus of points whose powers with respect to these circles are equal. Any two non-intersecting circles belong to a unique Apollonian family and, if the circles are not concentric, this locus is called the *radical axis* of the family or, equivalently, of any two circles in the family. Using this locus of equal powers definition, any two non-concentric circles have a radical axis: If the circles are tangent at a point then their common tangent line at that point is their radical axis, equivalently, it is the radical axis of the family of circles tangent at that point; if the circles intersect at two points then the line through these two points is their radical axis, equivalently, it is the radical axis of the family of circles that intersect at these two points. In each case, the radical axis can be thought of as the "circle of infinite radius" in the family.

Coaxal Circles and Radical Center

Circles that share a common radical axis are said to be *coaxal.* Thus, coaxal families of circles fall into three types: *Apollonian, the family of circles mutually tangent at a single point, the family of circles through two distinct points.* A family of concentric circles is actually an Apollonian family without a radical axis; it is determined by the common center and the Riemann/Möbius point ∞. The fact that it has no radical axis suggests that the radical axis of any coaxal family is an artifact of our Euclidean viewpoint, one that becomes irrelevant from the viewpoint of inversive geometry.

Now consider three non-coaxal circles. Each of the three pairs of circles has a radical axis, consisting of the points whose powers are equal with respect to both circles in that pair. But then a point of intersection of any two of these radical axes has the same power with respect to all three circles, so the third axis must also contain this point. Of course, two of the radical axes might be parallel, in which case all three are in the same parallel pencil since the radical axis of two circles is perpendicular to the line through their centers.

Theorem. *If the centers of three circles are not collinear then the three radical axes of the circles taken in pairs are concurrent.*

This point of concurrence is the unique point whose powers with respect to the three circles are equal. It is called the *radical center* of the three circles. The radical center is closely related to the orthocenter H of a triangle ABC. Let D be the foot of the altitude from A, let E be the foot of the altitude from B, and let F be the foot of the altitude from C. The right triangles AFH and CDH are similar and so $\angle FAH = \angle DCH$. Now D' be the other intersection of line AD with the circumcircle of ABC. Then $\angle B = \angle D'$ because both angles are subtended by the arc CA. But $\angle CHD = \angle B$ because $BDHF$ is a cyclic quadrilateral. Thus, the right triangles HDC and $D'DC$ are congruent; in particular, $HD = DD'$. Now the power of H with respect to the circumcircle is $(HA)\,(HD') = 2\,(HA)\,(HD)$. Similarly, the power of H is given by $2\,(HB)\,(HE)$ and $2\,(HC)\,(HF)$. It follows that, for any triangle ABC,

$$(HA)\,(HD) = (HB)\,(HE) = (HC)\,(HF)$$

Now let P be any point on BC. Since ADP is a right triangle the circle with diameter AP contains D. Similarly, the circles with diameters BQ and CR contain E and F, respectively, if Q is on CA and F is on AB. Thus:

Theorem. *The power of the orthocenter of a triangle with respect to any circle having a cevian as a diameter is half the power of the orthocenter with respect to the circumcircle.*

Corollary. *The orthocenter of a triangle is the radical center of any three circles whose diameters are cevians of the triangle.*

Morley's Theorem

Though it is elementary to state and prove, Morley's Theorem was not discovered until the twentieth century, probably because it involves angle trisectors which, in general, are not constructible with compass and straightedge:

Theorem. *The triangle determined by the points of intersection of the adjacent angle trisectors of any triangle is equilateral.*

There are now several proofs of Morley's Theorem but they all require the following observation. Consider four points Y', Z, Y, Z' such that $Y'Z = ZY = YZ'$ with $\angle YZY' = \angle Z'YZ$ and such that the bisectors of $\angle YZY'$ and $\angle Z'YZ$ are not parallel. Let O be the intersection of these bisectors. Then $OY'Z, OZY, OYZ'$ are isosceles triangles and so OY', OZ, OY, OZ' are radii of a circle with center O. The line $Y'Z'$ separates this circle into two arcs, the one containing Y, Z and the one in the other half-plane. Now suppose $\alpha < \frac{\pi}{3}$ and let the base angle of each of the isosceles triangles be $\frac{\pi}{2} - \alpha$. Then $\angle Y'OZ = \angle ZOY = \angle YOZ' = 2\alpha$, and so the arc of the circle not containing Y, Z is the locus of points in this half-plane from which the chord $Y'Z'$ subtends an angle 3α.

Proof of Morley's Theorem. In triangle ABC let $B = 3\beta$ and let $C = 3\gamma$ with their adjacent trisectors meeting at X. Let U be the intersection of their other two trisectors. Then BX bisects $\angle UBC$ and CX bisects $\angle UCB$, so X is the incenter of triangle BCU and therefore UX bisects $\angle CUB$. Let Y be on CU, in the same half-plane as C relative to XU, and let Z be on BU in the same half-plane as B relative to XU, such that $\angle ZXU = \frac{\pi}{6} = \angle YXU$. Then, since $\angle XUZ = \angle XUY$ we have $UXZ \simeq UXY$ and so $XXZ = XY$. But $\angle ZXY = \frac{\pi}{3}$, so XYZ is an equilateral triangle. If we can show that Z is the intersection of adjacent trisectors at A and B and Y is the intersection of adjacent trisectors at A and C then the theorem is proved. Note that $UX \perp ZY$ so UZY is isosceles with $UZ = UY$. Further, $\angle ZUY = \pi - 2(\beta + \gamma)$ so $\angle UYZ = \angle UZY = \beta + \gamma$. Now let $\angle BAC = 3\alpha$. Then $3\alpha + 3\beta + 3\gamma = \pi$ so $\beta + \gamma = \frac{\pi}{3} - \alpha$, therefore

$$\begin{aligned}
\angle XZU &= \frac{\pi}{3} + \angle UZY \\
&= \frac{\pi}{3} + \beta + \gamma \\
&= \frac{2\pi}{3} - \alpha \\
\angle UZY &= \frac{\pi}{3} - \alpha
\end{aligned}$$

Next, let Y' be on segment BA such that $BY' = BX$, and let Z' be on segment CA such that $CZ' = CX$. Then

$$\begin{aligned}
BZX &\simeq BZY' \\
CYX &\simeq CYZ'
\end{aligned}$$

Since XYZ is equilateral we have

$$Y'Z = ZY = YZ'$$

Further, $\angle UZY' = \angle XZU = \frac{2\pi}{3} - \alpha$ and $\angle UYZ' = \angle XYU = \frac{2\pi}{3} - \alpha$, since $\angle XZU = \angle XYU$. It follows that

$$\begin{aligned} \angle YZY' &= \angle YZU + \angle UZY' = \pi - 2\alpha \\ \angle Z'YZ &= \angle Z'YU + \angle UYZ = \pi - 2\alpha \end{aligned}$$

and $\alpha < \frac{\pi}{3}$ because $\angle BAC < \pi$. From the observation before the proof, A is on the arc of a circle through Y', Z, Y, Z' not containing Y, Z and the chord $Y'Z'$ subtends $\angle BAC = 3\alpha$. But then Z is the intersection of the adjacent trisectors at A and B and Y is the intersection of the adjacent trisectors at A and C. Thus the equilateral triangle XYZ is determined by the points of intersection of the adjacent angle trisectors of triangle ABC, which could have been any triangle. ■

Isometry and Congruence

A *transformation* of the Euclidean plane E^2 is a one-to-one and onto mapping of E^2 to itself, that is, a permutation of the points of E^2.

An *isometry* is a transformation of E^2 that preserves the absolute distance between points. Note that if T is an isometry then so is T^{-1}, and clearly the composition of two isometries is also an isometry. The group of isometries is denoted $E(2)$. If $\mathcal{F}$ and $\mathcal{F}'$ are figures in E^2 we say $\mathcal{F}$ and $\mathcal{F}'$ are congruent provided there is some $T \in E(2)$ such that $T(\mathcal{F}) = \mathcal{F}'$, whereby $T^{-1}(\mathcal{F}') = \mathcal{F}$.

A *collineation* is a transformation such that the image of any line of E^2 is also a line. The group of collineations is denoted $A(2)$.

Notation. For any point X let $\mathcal{C}_X(k)$ be the circle with center X and radius k.

Theorem. *$E(2)$ is a subgroup of $A(2)$.*

Proof. We must show that every isometry is a collineation. Let $T \in E(2)$ and let P, Q be distinct points. For any point X let $X' = T(X)$. Suppose R is on the line through P and Q, let p be the length of segment PR, and let q be the length of segment QR. Then $\mathcal{C}_P(p)$ and $\mathcal{C}_Q(q)$ are tangent at R. Since T is an isometry, R' must be on the circle $\mathcal{C}_{P'}(p)$ and also on the circle $\mathcal{C}_{Q'}(q)$, and since $PQ = P'Q'$ it follows that $\mathcal{C}_{P'}(p)$ and $\mathcal{C}_{Q'}(q)$ are tangent. Thus R' is the point of tangency and therefore on the line through P' and Q'. ■

Theorem. *An isometry is determined by its action on any three non-collinear points.*

Proof. Let $T \in E(2)$ and let A, B, C be non-collinear points, $T(A) = A', T(B) = B', T(C) = C'$. Given any point X we must show that $X' = T(X)$ is determined by A', B', C'. Let $a = AX, b = BX, c = CX$. Since T is an isometry X' must be on $\mathcal{C}_{A'}(a), \mathcal{C}_{B'}(b)$ and $\mathcal{C}_{C'}(c)$. Since A, B, C are non-collinear so are A', B', C' (otherwise T^{-1} would not be a collineation). Thus $\mathcal{C}_{A'}(a), \mathcal{C}_{B'}(b)$ and $\mathcal{C}_{C'}(c)$ have only X' in common. ■

In fact, any collineation is determined by its action on any three non-collinear points. This result is known as the **Fundamental Theorem of Affine Geometry** (FTAG) and it follows from the representation theorem for $A(2)$:

Theorem. *Let $T \in A(2)$ and for any point $P \in E^2$ represent P by the ordered pair (x, y) in R^2. Then*

$$T(P) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$$

for some choice of a, b, c, d, p, q with $ad - bc \neq 0$.

The isometries turn out to be those members of $A(2)$ for which $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. They are classified as follows.

I. Transformations generated by reflections

1. Definition. Let λ be a line and P be a point of the Euclidean plane. The point $P' = \lambda(P)$ is the *reflection* of P in λ provided λ is the $\perp$-bisector of the segment $\overline{PP}'$.

Note: Reflection interchanges the half-planes determined by λ, the *axis* of reflection, and the fixed points of this reflection are precisely the points on λ.

Theorem. *Every reflection is an isometry.*

Proof. Let λ be the axis of reflection, and let P, Q be distinct points, $P' = \lambda(P), Q' = \lambda(Q)$. Let M be the midpoint of $\overline{PP}'$ and let N be the midpoint of $\overline{QQ}'$. Then $P'MN$ and PMN are right triangles and hence congruent by **I.4** (SAS). Thus $P'N = PN$ and $\angle P'NM = \angle PNM$. But then $\angle P'NQ' = \angle PNQ$ and so triangles $P'NQ$ and PNQ are also congruent by **I.4**. Thus, $P'Q' = PQ$. ■

2. Composition of reflections in two lines λ and μ.

2.1 $\lambda = \mu$

In this case, $\lambda \circ \mu = \mu \circ \lambda = \iota$, the identity transformation.

2.2 $\lambda \cap \mu = \varnothing$

In this case, we have $\lambda \parallel \mu$ and we want to find $T(P)$ for, say, $T = \mu \circ \lambda$. Let d be the oriented distance from λ to μ measured along their common perpendicular ν through P, i.e., d is the distance from L to M, where $L = \lambda \cap \nu$ and $M = \mu \cap \nu$.Consider distances between points on ν to be signed numbers in accordance with $d > 0$. Let p be the distance from P to L. Thus the distance from P to $\lambda(P)$ is $2p$, whether p is positive or negative. We know that $T(P)$ is on ν and so its location will be determined provided we can specify its distance from P. First we note that the distance from P to M is $d + p$. Next, the distance from P to $\lambda(P)$ plus the distance from $\lambda(P)$ to M is the distance from P to M, and so the distance from $\lambda(P)$ to M is $d - p$. Finally, just as the distance from P to $\lambda(P)$ is $2p$, the distance from $\lambda(P)$ to $\mu \circ \lambda(P) = T(P)$ is $2(d - p)$. Thus, the distance from P to $T(P)$ is $2p + 2\,(d - p) = 2d$.

Conclusion: *The reflection in λ followed by the reflection in μ takes any point P to the point P' on ν such that the distance from P to P' is twice the distance from λ to μ. The composition $\mu \circ \lambda$ defines a translation, naturally associated with the vector of length d in the direction along ν from λ to μ, and this translation factors as $\beta \circ \alpha$, where α and β are any two lines parallel to λ with the distance from α to β equal to d. It also follows that if $T = \mu \circ \lambda$ then $T^{-1} = \lambda \circ \mu$. The identity ι can therefore be associated with the zero vector, whereby composition of translations corresponds to addition of vectors. In particular, composition of translations is commutative, even though composition of reflections is not. Note that a non-identity translation has no fixed point.*

2.3 $\lambda \cap \mu = Q$

In this case we want to find $T(P)$ for $T = \mu \circ \lambda$, relative to the point of intersection Q. By analogy to the distance between parallel lines we let θ be the angle from λ to μ measured in the counterclockwise direction, and take $\theta > 0$ by this convention. Given P, let $\mathcal{C}$ be the circle through P centered at Q and let L be the midpoint of the segment $\overline{P\lambda(P)}$. We will combine central angles of $\mathcal{C}$, observing the sign convention. First, $\measuredangle PQL = \measuredangle LQ\lambda(P)$, and so $\measuredangle PQ\lambda(P) = 2\measuredangle PQL$. Next let M be the midpoint of the segment $\overline{\lambda(P)T(P)}$. Then $\measuredangle \lambda(P)QM = \theta - \measuredangle LQ\lambda(P) = \measuredangle MQT(P)$. Thus, $\measuredangle PQT(P) = \measuredangle PQ\lambda(P) + \measuredangle \lambda(P)QT(P) = 2\measuredangle PQL + 2\measuredangle \lambda(P)QM = 2\theta$.

Conclusion: *The reflection in λ followed by the reflection in μ takes any point P to the point P' on C such $\measuredangle PQP' = 2\theta$. The composition $\mu \circ \lambda$ defines a rotation about the point Q with angular displacement θ, and this rotation factors as $\beta \circ \alpha$, where α and β are any two lines through Q with the angle from α to β equal to θ. A rotation with angular displacement $\theta = 0$ is the identity ι. It also follows that if $T = \mu \circ \lambda$ then $T^{-1} = \lambda \circ \mu$. Note that a non-identity rotation has a unique fixed point, which is the intersection Q of α and β. Any two rotations about Q commute with each other, though, as we will see, rotations about distinct centers do not.*

Suggested Exercises for Midterm

Draw two parallel lines λ and μ and a third line ν perpendicular to them. Let $L = \nu \cap \lambda$ and let $M = \nu \cap \mu$, and assume the distance from L to M is 2 units. Let P be the midpoint of the segment LM. Let $P' = \mu \circ \lambda(P)$.

a) Locate P'.
b) Determine the signed distance from M to P'.
c) Determine the signed distance from L to P'.

Let λ and μ be lines through point Q such that the angle from λ to μ is $\frac{\pi}{4}$. Let $T = \lambda \circ \mu$.

a) Determine the angle from λ to $\mu(\lambda)$.
b) Determine the angle from λ to $T(\lambda)$.

Draw two lines λ and μ through the point Q such that the angle from λ to μ is $\frac{\pi}{3}$. Locate a point P on the bisector of this angle such that the distance from Q to P is 4 units. Let $P' = \mu \circ \lambda(P)$.

a) Locate P'
b) Determine the angle from μ to the line through QP'.
c) Determine the angle from λ to the line through QP'.

Let $ABCD$ be a square. Label the vertices in this order counter-clockwise. Let T_1 be the reflection in line AB followed by the reflection in line AC, and let T_2 be the reflection in line BD followed by the reflection in line CD. Let $T = T_2 \circ T_1$.

a) Determine $T_1(B)$
b) Determine $T_2(A)$
c) Determine $T(X)$, where $X = AC \cap BD$
d) Determine $T(A), T(B), T(C), T(D)$
e) Let P be an arbitrary point in the plane. Explain how to find $T(P)$.

Let ABC be an isosceles triangle with angle B equal to angle C. Let β be the altitude cevian through B and let γ be the altitude cevian through C. In terms of the angle at A, find $\angle AHA'$, where $A' = \gamma \circ \beta(A)$.

Let ABC be an isosceles triangle with angle B equal to angle C. Let β be the internal angle bisector at B and let γ be the internal angle bisector at C. In terms of the angle at A, find $\angle AIA'$, where $A' = \gamma \circ \beta(A)$.

Let ABC be an isosceles triangle with angle B equal to angle C. Let β be the perpendicular bisector of AC and let γ be the perpendicular bisector of AB. In terms of the angle at A, find $\angle AOA'$, where $A' = \gamma \circ \beta(A)$.

Let ABC be an equilateral triangle. Describe the isometry that takes

a) triangle ABC to triangle ACB

b) triangle ABC to triangle BCA

c) triangle ABC to triangle CAB

d) triangle ABC to triangle CBA

e) triangle ABC to triangle BAC

f) triangle ABC to triangle ABC

Let $ABCD$ be a square. Label the vertices in this order counter-clockwise. For which permutations of the vertices does there exist an isometry that takes $ABCD$ to the permuted vertices? Describe the isometry in each of these cases. How many are there?

Let $ABCD$ be a rhombus. Label the vertices in this order counter-clockwise and let $P = AC \cap BD$. Describe the isometry that takes

a) triangle ABC to triangle ADC

b) triangle ABD to triangle CBD.

c) triangle APB to triangle CPB.

d) triangle APB to triangle APD.

e) triangle APB to triangle CPD.

f) If the rhombus is not a square, for which permutations of its vertices does there exist an isometry that takes $ABCD$ to the permuted vertices? Describe the isometry in each of these cases. How many are there?

In the coordinate plane, let λ be the line $y = x$ and let μ be the line $y = -x$. Let AOB be the triangle with $A = (1, 0)$, $O = (0, 0)$, and $B = (0, 1)$. Let $T = \mu \circ \lambda$, and for any point P let $P' = T(P)$.

a) Determine triangle $A'O'B'$.

b) Determine the image of AOB under T^{-1}.

c) Let ν be the line $y = x + 2$, and let $S = \nu \circ \lambda$. Determine the image of AOB under S.

d) Determine the image of AOB under S^{-1}.

3. Composition of three reflections

The associativity of composition allows us to determine two cases immediately. We assume three distinct lines κ, λ, μ.

3.1 $\kappa \parallel \lambda \parallel \mu$

We need to determine $T(P)$ where $T = \mu \circ \lambda \circ \kappa$. We can interpret T in two equivalent ways: the reflection κ followed by the translation $\mu \circ \lambda$; or the translation $\lambda \circ \kappa$ followed by the reflection μ. Choosing the first way, $T = (\mu \circ \lambda) \circ \kappa$, we let ν be the common perpendicular through P and let $K = \kappa \cap \nu$; let $L = \lambda \cap \nu$ and $M = \mu \cap \nu$, with d the distance from L to M. Let $P' = \kappa(P)$ and let $P'' = T(P)$. If the distance from P to K is p then $PP' = 2p$ and

$$\begin{aligned} PP'' &= PP' + P'P'' \\ &= 2p + 2d \\ &= 2\,(p + d) \end{aligned}$$

It follows that any point P moves along ν through the distance $2\,(p + d)$. However, if ξ is the line parallel to κ obtained by moving κ through the distance d then ξ will always be the $\perp$-bisector of segment PP''. Thus, $T(P) = \xi(P)$.

Conclusion: *The composition of reflections in three distinct parallel lines is a reflection in another line of their parallel pencil; in particular, that line is the translation of the first line of reflection through the distance from the second line to the third line.*

Now interpret T as the translation $\lambda \circ \kappa$ followed by the reflection μ, $T = \mu \circ (\lambda \circ \kappa)$. Let s be the distance from K to L and let p be the distance from P to M. Let $P' = (\lambda \circ \kappa)\,(P)$ and let $P'' = T(P)$. Then $PP' = 2s$ and

$$\begin{aligned} PP'' &= PP' + P'P'' \\ &= 2s + 2P'M \end{aligned}$$

Now $P'M = P'P + PM = -2s + p$, so

$$\begin{aligned} PP'' &= 2s + 2\,(-2s + p) \\ &= 2\,(p - s) \end{aligned}$$

It follows that any point P moves along ν through the distance $2\,(p - s)$. However, if ξ is the line parallel to μ obtained by moving μ through the distance $-s$ then ξ will always be the $\perp$-bisector of segment PP''. Thus, $T(P) = \xi(P)$.

Conclusion: *The composition of reflections in three distinct parallel lines is a reflection in another line of their parallel pencil; in particular, that line is the translation of the third line of reflection through the distance from the second line to the first line.*

Note: When either the first and second reflection, or the second and third reflection, are grouped together to produce a translation, their absolute locations are no longer relevant. All that matters is that they are members of the parallel pencil and that the oriented distance between them is preserved. This makes it easy to remember which line to move in order to obtain ξ: *Translate the remaining line of the pencil.*

3.2 $\kappa \cap \lambda \cap \mu = Q$

For three concurrent lines, suppose we choose to interpret $T = \mu \circ \lambda \circ \kappa$ as the reflection κ followed by the rotation $\mu \circ \lambda$, $T = (\mu \circ \lambda) \circ \kappa$. Given P, let $\mathcal{C}$ and Q be as in **2.3** with θ the angle from λ to μ. Let $P' = \kappa(P)$ and let K be the midpoint of the segment PP'. Let $P'' = T(P)$. If $\measuredangle PQK = \phi$ then $\measuredangle PQP' = 2\phi$ and $\measuredangle P'QP'' = 2\theta$. Thus, $\measuredangle PQP'' = 2\phi + 2\theta$. Let ξ be the line through Q that bisects $\measuredangle PQP''$. Then the angle from κ to ξ is θ, independent of P. Thus $T(P) = \xi(P)$.

Conclusion: *The composite of reflections in three distinct concurrent lines is a reflection in another line of their concurrent pencil; in particular, that line is the rotation of the first line of reflection through the angle from the second line to the third line.*

Exercise: Interpreting T as $\mu \circ (\lambda \circ \kappa)$, show that the line ξ is obtained by rotating μ through the angle from λ to κ. (See the note at the end of **3.1** above.)

We now consider three lines in general position:

3.3 $\mu \cap \lambda = Q$, where $Q \notin \kappa$

Let $T = \mu \circ \lambda \circ \kappa$ and let θ be the angle from λ to μ. By the note concluding **2.3**, the rotation $\mu \circ \lambda$ can be represented as the composition of reflections in any two lines through Q so long as the angle from the first to the second is θ. Therefore, if ϕ is the angle from κ to λ, rotate μ and λ about Q through the angle $-\phi$. Then μ and λ become μ' and λ', with $\lambda' \parallel \kappa$, and $T = \mu' \circ \lambda' \circ \kappa$, which we can interpret as the translation $\lambda' \circ \kappa$ followed by the reflection μ'. Let the perpendicular to κ through Q meet κ at K, and let d be the distance from K to Q. Now let α be the line through K that is parallel to μ'. It follows that the distance from α to μ' is $d \cos\theta$. Let β be the line through K that is perpendicular to α and let γ be the line through Q that is perpendicular to α. Then the distance from β to γ is $d \sin\theta$. Then $T = \gamma \circ \beta \circ \alpha$, which can be verified by determining the images of the three non-collinear points Q, K, and $L = \beta \cap \lambda$, noting in particular that $\kappa(Q) = \beta \circ \alpha(Q)$ and $\gamma(K) = \mu' \circ \lambda'(K)$. It follows that, for any point P, $\mu' \circ \lambda' \circ \kappa(P) = \gamma \circ \beta \circ \alpha(P)$.

Conclusion: *The composition of reflections in three lines in general position is a* ***glide****, that is, a reflection followed by a non-identity translation in a direction parallel to the line of the reflection. Specifically, if θ is the angle from the second line to the third line, Q is the intersection of these two lines, and K is the foot of the perpendicular through Q to the first line, then the reflection component of the glide is in the line obtained by rotating the first line through θ about K and the translation is through the distance $2d \sin\theta$, where $d = KQ$, in the direction parallel to the line of reflection; equivalently, the glide is the half-turn about K followed by reflection in the line through Q such that the angle from this line to the first line of reflection is the complement of θ.*

Notes:

1) In the first interpretation, from which the term **glide** is derived, the reflection and translation components of a glide commute; there is no fixed point but the line of the reflection component is an invariant line. In the second interpretation, the half-turn and reflection components do not commute. (How does switching the order change the motion?)

2) The translation component of a glide is in a direction parallel to the line of the reflection component. To determine which of the two parallel directions it is helpful to identify the translation component with a vector, as follows. For a given point P, let P' be its image after the reflection and let $P'' = T(P)$. The directed segment from P' to P'' can be represented as a vector by the directed line segment from K to Q' for some point Q'. Then P'' is the point such that Q' is in the same half-plane as Q relative to the first given line (the one that was rotated to obtain the invariant line of the glide). This is equivalent to the dot product of the vectors $\overrightarrow{KQ}$ and $\overrightarrow{P'P''}$ being positive.

4. Composition of finitely many reflections.

We now show that increasing the number of reflections results in no new type of transformation. First of all, if λ is an arbitrary line of reflection and T is a reflection, then $\lambda \circ T$ is either a translation or rotation, whereas if T is a rotation or translation, we have seen that $\lambda \circ T$ is either a reflection or a glide. Therefore It suffices to suppose that T is a glide and to analyze $\lambda \circ T$. We can assume that $T = \gamma \circ \beta \circ \alpha$, with $\alpha \perp \beta$ and $\beta \parallel \gamma$.

4.1 First suppose $\lambda = \alpha$. Then $\lambda \circ T = \alpha \circ (\gamma \circ \beta \circ \alpha) = \alpha \circ (\alpha \circ \gamma \circ \beta)$ since the translation and reflection components of a glide commute. But then $\lambda \circ T = \gamma \circ \beta$, a translation.

4.2 Next suppose that $\lambda \parallel \alpha$. Let $O = \alpha \cap \beta$ and let $Q = \gamma \cap \lambda$. Let $H_1 = \beta \circ \alpha$, the half-turn about O, and let $H_2 = \lambda \circ \gamma$, the half-turn about Q. If P is any point of the plane then $\lambda \circ T = (\lambda \circ \gamma) \circ (\beta \circ \alpha) = H_2 \circ H_1$, and so O is the midpoint of the segment $PH_1(P)$ and Q is the midpoint of the segment $H_1(P)P'$, where $P' = \lambda \circ T(P)$. Thus the line OQ is parallel to the line PP', and so triangle $OH_1(P)Q$ is similar to triangle $PH_1(P)P'$ with $PP' = 2OQ$. It follows that T is a translation parallel to OQ through twice the distance from O to Q. In other words, *the composite of half-turns about two distinct points is a translation through twice the distance between these points in the direction from the first point to the second.*

4.3 Finally, suppose $\lambda \cap \alpha = Q$. Since $\gamma \circ \beta$ is a translation we can assume that γ passes through Q. Now $\lambda \circ \gamma \circ \beta \circ \alpha = \lambda \circ \gamma \circ \alpha \circ \beta$ because $\beta \circ \alpha$ is a half-turn. By **3.2**, $\lambda \circ \gamma \circ \alpha$ is reflection in the line α' obtained by rotating α about Q through the angle from γ to λ, so a' is the line through Q that is perpendicular to λ. Thus $\lambda \circ T$ is a rotation about the point $\alpha' \cap \beta$. Specifically, *if the angle from α to λ is θ and the distance from β to γ is d, then $\lambda \circ T$ is rotation through 2θ about the intersection of β with the line through Q that is perpendicular to λ; in particular, this rotation is the half-turn about $\alpha \cap \beta$ if $\lambda \perp \alpha$.* **Exercise.** Show that the distance between Q and the center of this rotation is $d \csc \theta$.

Cases **4.1**, **4.2** and **4.3** exhaust the possibilities for λ and we have obtained the following classification of isometries generated by reflections.
Theorem. The composite of finitely many reflections is either a **reflection**, **translation**, **rotation** or **glide**. If the number of reflections is even (direct isometry) the result is either a translation or rotation; if the number of reflections is odd (opposite isometry) the result is either a reflection or glide.

II. Classification of isometries

The above theorem classifies the isometries of the plane that can be produced as a sequence of reflections. It remains to show that this process accounts for all isometries of the plane.

5. Congruent triangles.

Let ABC and $A'B'C'$ be distinct triangles with equal corresponding sides and angles. It is straightforward to show that there is a sequence of reflections that takes $A \mapsto A', B \mapsto B', C \mapsto C'$:

5.1 Same orientation

Suppose ABC and $A'B'C'$ have the same orientation. Let T be the translation such that $T(A) = A'$ and let R be the rotation about A' such that $R \circ T(B) = B'$. Then $R \circ T(C) = C'$ and so $R \circ T : ABC \mapsto A'B'C'$. Since $R \circ T$ is a direct isometry it can be represented as a composite of two reflections; in particular, ABC and $A'B'C'$ are related by a translation if two corresponding sides are parallel and by a rotation otherwise.

5.2 Opposite orientation

Suppose ABC and $A'B'C'$ have opposite orientation. If $A \neq A'$ let λ be the the $\perp$-bisector of the segment AA'. Then $\lambda(A)\lambda(B)\lambda(C) = A'\lambda(B)\lambda(C)$, and this reflected triangle has the same orientation as $A'B'C'$. Thus, there is a rotation R about A' such that $R \circ \lambda$ takes $B \mapsto B'$ and $C \mapsto C'$. It follows that ABC and $A'B'C'$ are related by either a reflection or a glide.

6. Uniqueness

We have seen that an isometry is determined by its action on three non-collinear points. Let us restate and prove this result in the present context.

Theorem. If ABC and $A'B'C'$ are triangles and T is an isometry such that $T(A) = A', T(B) = B', T(C) = C'$, then T is unique.

Proof. We must show that if P is any point of the plane then $P' = T(P)$ is determined by A', B' and C'. We will use the following proposition: $(*)$ If the intersection of two circles is the two distinct points X and Y then the $\perp$-bisector of the segment XY contains the centers of the circles. First note that by $(*)$ the circle centered at A of radius AP, the circle centered at B of radius BP, and the circle centered at C of radius CP intersect only at P (otherwise A, B, C would be collinear). Since the triangles ABC and $A'B'C'$ are related by the isometry their corresponding angles and sides are equal, and so there is a sequence of reflections that takes $A \mapsto A', B \mapsto B', C \mapsto C'$. Thus the circle centered at A' of radius AP, the circle centered at B' of radius BP, and the circle centered at C' of radius CP intersect in a unique point. But this point must be P' since $AP = A'P', BP = B'P', CP = C'P'$. Thus $T(P)$ is determined by $T(A), T(B), T(C)$. ∎

Main Theorem (Classification of Isometries). If T is an isometry of the plane then T is either a reflection, translation, rotation or glide. In particular, T can be represented as a composite of three or fewer reflections.

Proof. T is determined by its action on three non-collinear points. But by **5.1** and **5.2** this action can be realized as a finite sequence of reflections. ■

Suggested Exercises for Final Exam

Draw two parallel lines λ and μ and a third line κ perpendicular to them. Let $L = \kappa \cap \lambda$ and let $M = \kappa \cap \mu$, and let $LM = 2$ units. Let P be the midpoint of the segment LM.

a) Let $P' = \mu \circ \lambda(P)$. Locate P' and determine the signed distance MP'.

b) Let $P'' = \lambda \circ \mu \circ \lambda(P)$. Locate P'' and determine the signed distance $P''P$.

c) Find the line ν such that $P'' = \nu(P)$.

Draw two lines λ and μ through the point Q such that the angle from λ to μ is $\frac{\pi}{3}$. Locate a point P on the bisector of this angle such that the distance from Q to P is 4 units.

a) Let $P' = \mu \circ \lambda(P)$. Locate P' and determine the angle from μ to the line through QP'.

b) Let $P'' = \lambda \circ \mu \circ \lambda(P)$. Locate P'' and determine the angle from the ray QP to the ray QP''.

c) Find the line ν such that $P'' = \nu(P)$.

Let ABC be an equilateral triangle with side length 2 units, and let α, β, γ be the lines BC, CA, AB, respectively.

a) Find the invariant line of the isometry $T = \gamma \circ \beta \circ \alpha$.

b) Represent ABC in the coordinate plane so that the origin is the foot of the perpendicular from A and C is on the positive x-axis. Find the coordinates of $T(A), T(B), T(C)$.

In the coordinate plane, let κ be the x-axis, λ the y-axis, μ the line $x = 1$, and ν the line $y = 1$. Determine each of the following transformations T, and find the image of the triangle AOB, where $A = (1, 0)$, $O = (0, 0)$, and $B = (0, 1)$.

a) $T = \kappa \circ \lambda \circ \mu \circ \nu$

b) $T = \mu \circ \nu \circ \kappa \circ \lambda$

c) $T = \mu \circ \lambda \circ \kappa \circ \nu$

d) $T = \nu \circ \kappa \circ \lambda \circ \mu$

Let P and Q be two points. Let R be the rotation about P through $\frac{\pi}{4}$ and let T be the translation from P to Q.

a) Show that the isometry $S = T \circ R$ is a rotation.

b) Where is the center of the rotation S ? Note that $S : PQ \mapsto QQ'$.

c) Generalize if R is the rotation about P through θ.

Let P be a point on the line λ and let R be the rotation through θ about P.

a) Determine the isometry $T = R \circ \lambda$.

b) Determine the isometry $\lambda \circ T$.

Label the vertices of triangle ABC in order counter-clockwise and measure the internal angles A, B, C counter-clockwise. Let R_V be the rotation about vertex V through twice the internal angle at the vertex.

a) Describe R_A, R_B, R_C in terms of the reflections $\alpha = BC, \beta = CA, \gamma = AB$.

b) What is the isometry $R_B \circ R_C \circ R_A$?

For triangle ABC let α, β, γ be the $\perp$-bisectors of BC, CA, AB, respectively. Determine the lines λ, μ, ν such that

a) $\lambda = \gamma \circ \beta \circ \alpha$

b) $\mu = \beta \circ \alpha \circ \gamma$

c) $\nu = \alpha \circ \gamma \circ \beta$

d) There are six ways to order the reflections α, β, γ. What happens with the other three orders of composition?

Let P and Q be distinct points on the line λ. Let R_P be the half-turn about P and let R_Q be the half-turn about Q.

a) Describe the isometry $T = R_Q \circ \lambda \circ R_P$.

b) Find the invariant line of T.

c) Locate $T(M)$, where M is the midpoint of segment PQ.

Determine the composition of reflections in the three angle bisectors of a triangle, which will be a reflection in some line through the incenter.

Let ABC be an isosceles triangle with $AB = AC$ and angle $A = \frac{\pi}{6}$. Let α, β, γ be the altitude cevians at A, B, C, respectively.

a) Find λ such that $\lambda = \gamma \circ \beta \circ \alpha$.

b) Let λ intersect BC at P. Find $\angle HCP$.

c) For what angle at A would λ be parallel to BC ?

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles that intersect at two points X and Y. Let H be the half-turn about X. Then $\mathcal{C}_2$ intersects $H(\mathcal{C}_1)$ at X and another point Z.

a) Locate $U = H(Z)$.

b) Locate $V = H(Y)$.

c) What type of quadrilateral is $YUVZ$?