Math 331 Handout # 1 Winter, 2012

1. Let V be a vector space and $u, v, w \in V$ such that $u + v = u + w$. Then prove that $v = w$.
$u + v = u + w \implies -u + u + v = -u + u + w$(since $-u \in V$) $\implies (-u + u) + v = (-u + u) + w$ (since associative law under addition holds in V) $\implies 0 + v = 0 + w$ (since $-u$ is the additive inverse of $u \in V$) $\implies v = w$(since 0 is the zero vector of V).

2. Let v be any element of a vector space V, and let c be any scalar. Then
(a) $0v = 0_V$
$0v = (0 + 0)v = 0v + 0v$. $0v = 0v + 0v \implies 0_V + 0v = 0v + 0v \implies 0_V = 0v$ (by 1 above).
(b) $c0_V = 0_V$
$c0_V = c(0_V + 0_V) = c0_V + c0_V$. $c0_V = c0_V + c0_v \implies 0_V = c0_V$ (as above)
(c) If $cv = 0_V$, then c=0 or v=0_V.
Assume $c \neq 0$. Then $\frac{1}{c}(cv) = \frac{1}{c}(0_V) \implies$ v=0_V.
(d) $(-1)v = -v \; \forall v \in V$.
Let $v \in V$. Prove: $(-1)v = -v$
$(-1)v + v = (-1)v + 1v$ (since $1v = v \; \forall v \in V$) $= (-1 + 1)v$ (since $(c + d)v = cv + dv \; \forall c, d \in \mathbb{R}$)
0v = 0 (since $0v = 0 \forall v \in V$). Hence $(-1)v + v = 0. \implies (-1)v = -v$.

3. Prove that every vector space has exactly one zero vector. Let $0, 0'$ be zero vectors of V. Prove $0 = 0'$.
$0 + 0' = 0$ (since 0 is a vector of V and $0'$ is a zero vector of V). $0 + 0' = 0'$ (since $0'$ is a vector of V and 0 is a zero vector of V). Thus $0 = 0 + 0' = 0'$. Hence $0 = 0'$.

4. Prove that in a vector space V, the additive inverse of a vector is unique. Let $v \in V$ and let $-v$ and u be its inverses. Prove $-v = u$.
Now $v + (-v) = 0_V$ and $v + u = 0_V$. Thus $v + (-v) = v + u$. Hence $u = -v$ (by 1 above).

5. If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.
We apply the "subspace criterion" to $V \cap W$.
1. $0_U \in V$ and $0_U \in W$, since V and W are subspaces of U. Hence $0_U \in V \cap W$.
2. Let $u, v \in V \cap W$. Then $u, v \in V$ and $u, v \in W$ (definition of intersection). Now $u, v \in V \implies u + v \in V$ (axiom 1 (since V is a subspace and therefore a vectorspace)) and $u, v \in W \implies u + v \in W$ (axiom 1 (since W is a subspace and therefore a vectorspace)). Thus $u + v \in V \cap W$.
3. Let $c \in \mathbb{R}$ and $u \in V \cap W$. Then $u \in V$ and $u \in W$ (definition of intersection). Now $u \in V$ and $c \in \mathbb{R} \implies cu \in V$ (axiom 6 (since V is a subspace and therefore a vectorspace)) and $u \in W$ and $c \in \mathbb{R} \implies cu \in W$ (axiom 6 (since W is a subspace and therefore a vectorspace)). Thus $cu \in V \cap W$.

6. A set S = $\{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly dependent if and only if one of the vectors v_j is a linear combination of the other vectors in S.
($\implies$): Assume S = $\{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly dependent. $\implies \exists \; c_1, c_2, \ldots, c_k$, not all zero, such that $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0_V$. Assume $c_1 \neq 0$. Then $v_1 = (-\frac{c_2}{c_1})v_2 + \ldots + (-\frac{c_k}{c_1})v_k$
($\impliedby$): Assume $v_1 = c_2v_2 + c_3v_3 + \ldots + c_kv_k$. Then $(-1)v_1 + c_2v_2 + c_3v_3 + \ldots + c_kv_k = 0_V$. Hence S = $\{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly dependent.

7. Two vectors u and v in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.
($\implies$): Assume u and v are linearly dependent. Then $\exists \; c_1, c_2 \in \mathbb{R}$, not both zero, such that $c_1u + c_2v = 0_V$. Assume $c_1 \neq 0$. Then $u = (-\frac{c_2}{c_1})v$.
($\impliedby$): Assume $u = cv$, for some $c \in \mathbb{R}$. Then $(-1)u + cv = 0_V$. Thus u and v are linearly dependent.

8. Let S and S' be subsets of the vector space V such that $S' \subseteq S$, say $S = \{v_1, v_2, v_3, v_4, v_5\}$ and $S' = \{v_1, v_2, v_3\}$.

   (a) If S is linearly independent, then S' is linearly independent.
   Assume S' is linearly dependent. Thus *exists* $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that $c_1v_1 + c_2v_2 + c_3v_3 = 0_v$. $\implies$ *exists* $c_1, c_2, c_3, a_4, a_5 \in \mathbb{R}$, not all zero, such that $c_1v_1 + c_2v_2 + c_3v_3 + a_4v_4 + a_5v_5 = 0_v$. Hence S is linearly dependent, a contradiction to the given.

   (b) If S' is linearly dependent, then S is linearly dependent.
   The required statement is equivalent to "If S is linearly independent, then S' is linearly independent".

9. Given that $\{u_1, u_2, \dots, u_n\}$ is a linearly independent set of vectors and that the set $\{u_1, u_2, \dots, u_n, v\}$ is linearly dependent, then v is a linear combination of the u_i's.
$\{u_1, u_2, \dots, u_n, v\}$ is linearly dependent $\implies \exists c_1, c_2, \dots, c_n, c_{n+1}$, not all zero, such that $c_1u_1 + c_2u_2 + \dots + c_nu_n + c_{n+1}v = 0_V$. If $c_{n+1} = o$, then $\{u_1, u_2, \dots, u_n\}$ is a linearly dependent, a contradiction. Thus $c_{n+1} \neq 0$. Hence $-c_{n+1}v = c_1u_1 + c_2u_2 + \dots + c_nu_n$ and $v = (-\frac{c_1}{c_{n+1}})u_1 + (-\frac{c_2}{c_{n+1}})u_2 + \dots + (-\frac{c_n}{c_{n+1}})u_n$.

10. If the vector space V is spanned by $\{v_1, v_2, \dots, v_k\}$ and one of these vectors can be written as a linear combination of the other k-1 vectors, then prove that the span of these k-1 vectors is also V.

11. $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V if and only if every vector v in V can be written in exactly one way as a linear combination of vectors in S.
($\Longrightarrow$): Assume $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$ and $v = d_1v_1 + d_2v_2 + \dots + d_nv_n$ Then $(c_1 - d_1)v_1 + (c_2 - d_2)v_2 + \dots + (c_n - d_n)v_n = 0_V$. Since $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent, $(c_i - d_i) = 0$, for each i, $1 \leq i \leq n$ and so $c_i = d_i$, for each i, $1 \leq i \leq n$.
($\Longleftarrow$): Assume every vector v in V can be written in exactly one way as a linear combination of vectors in S. Then $S = \{v_1, v_2, \dots, v_n\}$ spans V, and S is linearly independent since $0v_1 + 0v_2 + \dots + 0v_n = 0_V = c_1v_1 + c_2v_2 + \dots + cv_n \implies c_i = 0$, for each i, $1 \leq i \leq n$.

12. If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.
The above statement is equivalent to: Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V. Then every linearly independent set of vectors in V has less than or equal to n vectors.

13. If a vector space V has one basis with n vectors, then every basis for V has n vectors.
Let B and B′ be bases of V with m and n vectors, respectively. Prove m=n. Since B is a basis and B′ is linearly independent, n ≤ m. Since B′ is a basis and B is linearly independent, m ≤ n.

14. Let V be a vector space of dimension n.

   (a) If $S = \{v_1, v_2, \dots, v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
   S is a subset of a basis B of V (see 18 below). Since $|B| = dimV = n$, S=B. Thus S is a basis for V.

   (b) If $S = \{v_1, v_2, \dots, v_n\}$ spans V, then S is a basis for V.
   S contains a basis B of V (see 17 below). Since $|S| = |B| = n$, S=B. Thus S is a basis for V.

15. If V is a vector space with dim(V) = n, then any set of n+1 vectors in V is linearly dependent.
By 14(a) above, if $S = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V. Then every linearly independent set of vectors in V has less than or equal to n vectors.

16. If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V and c is a nonzero real number, then $S_1 = \{cv_1, cv_2, \dots, cv_n\}$ is also a basis for V.
Since dim(V) = n and S_1 has n vectors, it suffices to show that S_1 is linearly independent. Assume $c_1(cv_1) + c_1(cv_2) + \dots + c_n(cv_n) = 0_V$. Then $(c_1c)v_1 + (c_1c)v_2 + \dots + (c_nc)v_n = 0_V$. Since S is linearly independent, $cc_i = 0$, for each i, $1 \leq i \leq n$. Thus $c_i = 0$, for each i, $1 \leq i \leq n$, since $c \neq 0$.

17. Let S be a spanning set for the finite dimensional vector space V. Then there exists a subset S' of S that forms a basis for V. That is: Every spanning set contains a basis.

18. Let S be a linearly independent set of vectors from the finite dimensional vector space V. Then there exists a basis for V containing S.
    That is: Every linearly independent set can be extended to give a basis.

19. Let V be a vector space of dimension n. Then any set of less than n vectors cannot span V.
    Since a set of less than n vectors cannot contain a basis of V with n vectors (dim V = n).

20. If A is an $n \times n$ matrix, then the following conditions are equivalent.
    1. A is invertible.
    2. Ax=b has a unique solution for any $n \times 1$ matrix b.
    3. Ax=0 has only the trivial solution.
    4. A is row-equivalent to I_n.
    5. $|A| \neq 0$.
    6. Rank(A)=n.
    7. The n row vectors of A are linearly independent.
    8. The n column vectors of A are linearly independent.