Math 553 TEST #2 Material Fall 2009 (Test Date: November 19, 2009)

`Homeworks #3, #4, #5 Chapter 2 Section 2.3 Theorems: 2.32, 2.3.7 Exercises 2.3 (page 65): Problem: 1, 17 Section 2.4 Theorems: 2.4.3; 2.4.7 (a), (b) Corollay: 2.4.8 Examples: 2.4.2(a),(b); 2.4.4 (a) Exercises 2.4 (page 72): Problem: 1; 2 (a); 3; 7(a)-(f); 8 Section 2.6 Theorems: 2.6.2 (a), (b); 2.6.3; 2.6.4 Exercises 2.6 (page 85): Problem: 1; 2 (a)-(e) Chapter 3 Section 3.1 Theorems: 3.1.5; 3.1.6 (a), (b); 3.1.7 (a), (b); 3.1.9 Examples: 3.12 (a), (b); 3.14 (a)-(e); 3.1.8 (a) Exercises 3.1 (page 100): Problem: 1-3`

1. If $\{a_n\}$ is bounded and increasing then $\{a_n\}$ converges.
   $\{a_n\}$ is bounded $\implies A = \{a_n : n \in \mathbb{N}\}$ is bounded above and nonempty. Let $\alpha = \sup A$. Let $\epsilon > 0$. Then $\exists\, a_{n_0} \in A$ such that $\alpha - \epsilon < a_{n_0}$. $\implies \alpha - \epsilon < a_{n_0} \leq a_n \leq \alpha\ \forall n \geq n_0$.

2. If $\{a_n\}$ is bounded and decreasing then $\{a_n\}$ converges.
   $\{a_n\}$ is bounded $\implies A = \{a_n : n \in \mathbb{N}\}$ is bounded below and nonempty. Let $\beta = \inf A$. Let $\epsilon > 0$. Then $\exists\, a_{n_0} \in A$ such that $a_{n_0} < \beta + \epsilon \implies a_n \leq a_{n_0} < \beta + \epsilon\ \forall n \geq n_0$.

3. If the sequence $\{a_n\}$ is increasing and not bounded above, then prove that $\{a_n\}$ diverges to ∞.
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $a_n > M$.
   $\{a_n\}$ is unbounded $\implies A = \{a_n : n \in \mathbb{N}\}$ is unbounded above. Thus, $\exists$ an $n_0 \in \mathbb{N}$ such that $a_{n_0} \in A$ and $a_{n_0} > M$. Now $\{a_n\}$ is increasing $\implies a_n \geq a_{n_0}\ \forall n \geq n_0$. Hence, if $n \geq n_0$ then $a_n > M$. So $a_n \longrightarrow \infty$.

4. If $\{n_k\}$ is a strictly increasing sequence of positive integers then $n_k \geq k\ \forall k \in \mathbb{N}$.
   Prove by induction: (1) $n_1 \geq 1$, by definition. (2) Assume $n_k \geq k$. (3) $n_{k+1} > n_k$, by definition of n_k. By (2) $n_k \geq k$. Thus $n_{k+1} > n_k \geq k$. So $n_{k+1} > k$. Hence $n_{k+1} \geq k+1$.

5. If the sequence $\{a_n\}$ converges to a, then every subsequence of $\{a_n\}$ also converges to a.
   Let $\epsilon > 0$. Find $k_0 \in \mathbb{N}$ such that $|a_{n_k} - a| < \epsilon\ \forall k \geq k_0$. $a_n \longrightarrow a \implies \exists n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon\ \forall n \geq n_0$. Choose $k_0 = n_0$. If $k \geq k_0 = n_0$, then (since $n_k \geq k \geq k_0 = n_0$) $|a_{n_k} - a| < \epsilon$.

6. Every sequence has a monotonic subsequence.
   Let $\{a_n\}$ be a sequence. Let S be the set of all positive integers n such that a_n is a lower bound for the set $\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\}$. If S is infinite, then S can be expressed as a strictly increasing sequence $\{n_k\}$ and the subsequence $\{a_{n_k}\}$ is increasing. If S is finite, then there exists an integer n_0 larger than every element in S. Let n_1 be larger than n_0. Since $n_1 \notin S$, a_{n_1} is not a lower bound for the set $\{a_{n_1+1}, a_{n_1+2}, a_{n_1+3}, \ldots\}$, so there exists an integer $n_2 > n_1$ such that $a_{n_2} < a_{n_1}$. Similarly, there exists an integer $n_3 > n_2$ such that $a_{n_3} < a_{n_2}$. Continuing this process yields a decreasing sequence $\{a_{n_k}\}$.

7. Bolzano-Weierstrass: Every bounded sequence has a convergent subsequence.
   Let $\{a_n\}$ be a bounded sequence. So $\exists\, M$ such that $|a_n| \leq M\ \forall n \in \mathbb{N}$. Also, $\{a_n\}$ has a monotonic subsequence, say $\{a_{n_k}\}$ (by 2 above). Then $|a_{n_k}| \leq M\ \forall k \in \mathbb{N}$ (since $a_{n_k},\ k \in \mathbb{N}$ are some of the terms of the sequence $\{a_n\}$ and $|a_n| \leq M\ \forall n \in \mathbb{N}$). Thus $\{a_{n_k}\}$ is bounded and monotonic and therefore converges (since "every bounded and monotonic sequence converges"). Hence $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$.

8. Every convergent sequence is a Cauchy sequence.
   Let $\epsilon > 0$. $\{a_n\}$ converges to $a \implies \exists n_0 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2} \forall n \geq n_0$. Then, if $m, n \geq n_0$, then $|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| = |a_n - a| + |a_m - a| = |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

9. Every Cauchy sequence is bounded.
$\{a_n\}$ is Cauchy $\implies \exists\ n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1\ \forall m, n \geq n_0$. So $|a_n| - |a_{n_0}| \leq |a_n - a_{n_0}| < 1 \forall n \geq n_0 \implies |a_n| < 1 + |a_{n_0}| \forall n \geq n_0$. Define M =Max$\{|a_1|, |a_2|, \ldots, |a_{n_0}|, 1 + |a_{n_0}|\}$. Thus $|a_n| \leq M\ \forall n \in \mathbb{N}$.

10. If a Cauchy sequence $\{a_n\}$ has a convergent subsequence, then $\{a_n\}$ converges.
Let $\{a_{n_k}\}$ be a subsequence of the Cauchy sequence $\{a_n\}$ and let $\{a_{n_k}\}$ converge to a. Let $\epsilon > 0$. Since $\{a_n\}$ is Cauchy, $\exists n_1 \in \mathbb{N}$ such that $|a_n - p_m| < \frac{\epsilon}{2} \forall m, n \geq n_1$ and since $\{a_{n_k}\}$ converges to a, $\exists k_1 \in \mathbb{N}\ |a_{n_k} - a| < \frac{\epsilon}{2} \forall k \geq k_1$ (Note that $|a_{n_k} - a| < \frac{\epsilon}{2}$ is satisfied by all $n_k \in \{n_{k_1}, n_{k_1+1}, n_{k_1+2}, \ldots\}$.). Choose $n_0 =$ Max $\{n_1, k_1\}$. If $n \geq n_0$, (since $n_{n_0} \geq n_0 \geq n_1$ and $n_0 \geq k_1$), then $|a_n - a| \leq |a_n - a_{n_{n_0}}| + |a_{n_{n_0}} - a| < \epsilon$.

11. Every Cauchy sequence converges.
Let $\{a_n\}$ be a Cauchy sequence. Then $\{a_n\}$ is bounded, by 9 above. $\implies$ $\{a_n\}$ has a convergent subsequence (Bolzano-Weierstrass). $\implies$ $\{a_n\}$ converges, by 10 above.

12. Theorem 2.4.7: Let $E \subset \mathbb{R}$.
(a) If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.
Assume there is a nhd M of p with $M \cap E \setminus \{p\} = \{q_1, q_2, \ldots, q_n\}$. Take $\epsilon =$ Min $\{|q_1 - p|, |q_2 - p|, \ldots, |q_n - p|\}$. Then $\epsilon > 0$ and $N_\epsilon(p) \cap E \setminus \{p\} = \emptyset$ (since $r \in N_\epsilon(p) \implies |r - p| < \epsilon \leq |q_i - p| \implies r \neq q_i$). $\implies$ p is not a limit point of E.
(b) If p is a limit point of E, then there exists a sequence $\{p_n\}$ in E, with $p_n \neq p$ for all $n \in \mathbb{N}$, such that $\underset{n\to\infty}{lim} p_n = p$.
p is a limit point of $E \implies \forall n \in \mathbb{N}\ \exists p_n \in E \setminus \{p\}$ such that $|p_n - p| < \frac{1}{n}$.

13. Corollary 2.4.8: A finite set has no limit points.
Let A be a finite set and $p \in \mathbb{R}$. Then no nhd of p contains infinitely many points of A. So $\exists$ a nhd of p that does not contain infinitely many points of A.

14. Every $\epsilon-$neighborhood is open.
Let $p \in \mathbb{R}$ and let $\epsilon > 0$. Show that $N_\epsilon(p)$ is open. Let $q \in N_\epsilon(p)$. Define $\delta = \epsilon - |q - p|$. Note $\delta > 0$ since $q \in N_\epsilon(p) \implies |q - p| < \epsilon$. Now prove $N_\delta(q) \subseteq N_\epsilon(p)$. Let $x \in N_\delta(q)$. Then $|x - p| = |x - q + q - p| \leq |x - q| + |q - p| < \delta + |q - p| = \delta + (\epsilon - \delta) = \epsilon$. So $x \in N_\epsilon(p)$.

15. Theorem 3.1.5: Every open interval in $\mathbb{R}$ is an open set.
Open Intervals in $\mathbb{R}$ are: (a, b), (a, ∞), $(-\infty, a)$, and $(-\infty, \infty)$.
open subset of $\mathbb{R}$.
Open Intervals are: (a, b), (a, ∞), $(-\infty, a)$, and $(-\infty, \infty)$.
Prove : $(-\infty, \infty)$ is open: Let $p \in (-\infty, \infty)$. Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) \subseteq (-\infty, \infty)\ \forall \epsilon > 0$.
Prove (a, b) is open: Let $p \in (a, b)$. Choose $\epsilon =$ Min $\{p - a, b - p\}$. Then $a \leq p - \epsilon < p < p + \epsilon \leq p + b - p = b$. Thus $(p - \epsilon, p + \epsilon) \subseteq (a, b)$.
Prove (a, ∞) is open .
Let $p \in (a, \infty)$. Find an $\epsilon > 0$ such that $N_\epsilon(p) \subset (a, \infty)$. Choose $\epsilon = p - a$. Then $\epsilon > 0$ (since $p \in (a, \infty) \implies p > a \implies p - a > 0$). Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) = (p - (p - a), p + (p - a)) = (a, 2p - a) \subset (a, \infty)$. (Note that any ϵ such that $0 < \epsilon < p - a$ will also work). So (a, ∞) is open.

Prove that $(-\infty, a)$ is open:
Let $p \in (-\infty, a)$. Find an $\epsilon > 0$ such that $N_\epsilon(p) \subset (-\infty, a)$. Choose $\epsilon = a - p$. Then $\epsilon > 0$ (since $p \in (-\infty, a) \implies p < a \implies a - p > 0$). Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) = (p - (a - p), p + (a - p)) = (2p - a, a) \subset (-\infty, a)$. (Note that any ϵ such that $0 < \epsilon < a - p$ will also work). So $(-\infty, a)$ is open.

16. Theorem 3.1.6 (a) For any collection $\{O_\alpha : \alpha \in I\}$ of open subsets of $\mathbb{R}$, $\cup_{\alpha \in I} O_\alpha$ is open.
Let $p \in \cup_{\alpha \in I} O_\alpha$. Then $p \in O_\alpha$ for some $\alpha \in I$ and O_α is open.
(b) For any finite collection $\{O_1, O_2, \ldots, O_n\}$ of open subsets of $\mathbb{R}$, $\cap_{i=1}^n O_i$ is open.

Let $p \in \cap_{i=1}^n O_i$. Then $p \in O_i \forall i \in \{1, 2, \ldots, n\}$. $\implies \exists \epsilon_i > 0$ such that $N_{\epsilon_i}(p) \subset O_i \forall i \in \{1, 2, \ldots, n\}$. Choose $\epsilon =$ Min $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $N_\epsilon(p) \subset O_i \forall i \in \{1, 2, \ldots, n\}$ (since $q \in N_\epsilon(p) \implies |q - p| < \epsilon \leq \epsilon_i \implies q \in N_{\epsilon_i}(p) \subset O_i$) $\implies N_\epsilon(p) \subset \cap_{i=1}^n O_i$.

17. Every closed interval in $\mathbb{R}$ is a closed set.

    $[a, \infty)^c = (-\infty, a)$, an open set (as shown above). Thus the complement of $[a, \infty)$ is open. Hence $[a, \infty)$ is closed (by definition of a closed set).
    $(-\infty, a]^c = (a, \infty)$, an open set (as shown above). Thus the complement of $(-\infty, a]$ is open. Hence $(-\infty, a]$ is closed (by definition of a closed set).
    $[a, b]^c = (-\infty, a) \cup (b, \infty)$, a union of two open sets (as shown above) and therefore open. Thus the complement of $[a, b]$ is open. Hence $[a, b]$ is closed (by definition of a closed set).

18. Theorem 3.1.7: (a) For any collection $\{F_\alpha : \alpha \in I\}$ of closed subsets of $\mathbb{R}$, $\cap_{\alpha \in I} F_\alpha$ is closed.
    $(\cap_{\alpha \in I} F_\alpha)^c = \cup_{\alpha \in I} F_\alpha^c$. Now apply Theorem 3.1.6(a).
    (b) For any finite collection $\{F_1, F_2, \ldots, F_n\}$ of closed subsets of $\mathbb{R}$, $\cup_{i=1}^n F_i$ is closed.
    $(\cup_{i=1}^n F_i)^c = \cap_{i=1}^n F_i^c$. Now apply Theorem 3.1.6(b).

19. Theorem 3.1.9: A subset F of $\mathbb{R}$ is closed if and only if F contains all its limit points.
    ($\Longrightarrow$) Assume F is closed. Let p be a limit point of F. Show $p \in F$. Suppose $p \notin F \implies p \in F^c$. $\implies \exists\ \epsilon > 0$ such that $N_\epsilon(p) \subset F^c$ (since F^c is open). $\implies p$ is not a limit point of F.
    ($\Longleftarrow$) Assume F contains all its limit points. In order to show F is closed, prove F^c is open. Let $p \in F^c$. Then p is not a limit point of F (since F contains all its limit points). $\implies \exists \epsilon > 0$ such that $N_\epsilon(p) \setminus \{p\} \cap F = \emptyset$.

20. The set of limit points of a set is closed.
    Let $E \subset \mathbb{R}$ and E' be the set of limit points of E. We need to prove that E' is closed. So we will show that E' contains all of its limit points ($F \subset \mathbb{R}$ is closed $\iff F$ contains all of its limit points). Let p be a limit point of E'. Prove: $p \in E'$. We will show that p is a limit point of E (since $p \in E' \iff p$ is a limit point of E). Let $\epsilon > 0$ and prove $N_\epsilon(p) \cap E \setminus \{p\} \neq \emptyset$; that is, E contains a point of $N_\epsilon(p)$, other than p. Now $N_\epsilon(p) \cap E' \setminus \{p\} \neq \emptyset$ (since p is a limit point of E'). Thus $\exists q \in N_\epsilon(p) \cap E' \setminus \{p\} \implies q \in N_\epsilon(p)$ and $q \in E' \setminus \{p\}$. $q \in E' \setminus p \implies q$ is a limit point of E(definition of E') and $q \in N_\epsilon(p) \implies \exists$ a $\delta > 0$ such that $N_\delta(q) \subset N_\epsilon(p)$ (since $N_\epsilon(p) = (p - \epsilon, p + \epsilon)$ is open). Since q is a limit point of E, $N_\delta(q)$ has infinitely many points of E and hence a point of E, other than p. Thus $N_\epsilon(p)$, (since $N_\delta(q) \subset N_\epsilon(p)$), has a point of E, other than p.