Math 553 TEST #1 Material Fall 2009 (Test Date: October 20, 2009)

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1. Theorem: 1.4.4: Let $A$ be a nonempty subset of $\mathbb{R}$ that is bounded above. An upper bound $\alpha$ is the supremum of $A$ if and only if for every $\beta < \alpha$, $\exists$ an element $x \in A$ such that $\beta < x \leq \alpha$.
   ($\Longrightarrow$): $\alpha = \sup A$. $\beta < \alpha \implies \beta$ is not an upper bound of $A$.
   ($\Longleftarrow$) $\alpha$ is an upper bound of $A$. Let $\beta < \alpha$ then $\exists$ an element $x \in A$ such that $\beta < x < \alpha \implies \beta$ is not an upper bound of $A$.

2. Lemma 1: $\mathbb{N}$ is not bounded above.
   $\alpha = \text{lub } \mathbb{N} \implies \exists$ an $n \in \mathbb{N}$ with $\alpha - 1 < n$.

3. Lemma 2: For every $\epsilon > 0$, there is an n in $\mathbb{N}$ with $\frac{1}{n} < \epsilon$.
   $\frac{1}{\epsilon}$ is not an upper bound for $A$. $\implies \exists\ n \in \mathbb{N}$ such that $\frac{1}{\epsilon} < n$.

4. Theorem 1.5.1: If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists$ a positive integer n such that $nx > y$.
   $\frac{y}{x}$ is not an upper bound of $\mathbb{N} \implies \exists$ an $n \in \mathbb{N}$ such that $\frac{y}{x} < n$.

5. Theorem 1.5.2: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\ r \in \mathbb{Q}$ such that $x < r < y$.
   Assume $x \geq 0$. Then $y - x > 0 \implies \exists n \in \mathbb{N}\ n(y - x) > 1 \implies ny > 1 + nx$. Now $\{k \in \mathbb{N} : nx < k\} \neq \emptyset$ has a least element $m$. So $m - 1 \leq nx < m$. Thus $m \leq 1 + nx < ny$. Also, $nx < m$.

6. Theorem 2.1.10:
   (a)If a sequence $\{a_n\}$ converges then its limit is unique.
   Assume $\{a_n\}$ converges to $l$ as well as to $m$. Let $\epsilon > 0$ be given. Then $\exists\ n_1 \in \mathbb{N}$ such that $|a_n - l| < \frac{\epsilon}{2} \forall n \geq n_1$ and $\exists\ n_2 \in \mathbb{N}$ such that $|a_n - m| < \frac{\epsilon}{2} \forall n \geq n_2$. Choose $n_0 = Max\{n_1, n_2\}$. Then $|l - m| \leq |a_{n_0} - l| + |a_{n_0} - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(b) If a sequence $\{a_n\}$ converges then $\{a_n\}$ is bounded.
Assume $\{a_n\}$ converges to $a$. Then $\exists n_0 \in \mathbb{N}$ such that $|a_n - a| < 1 \forall n \geq n_0$. Now $|a_n| - |a| \leq |a_n - a| < 1 \forall n \geq n_0$. Thus $|a_n| \leq (1 + |a|) \forall n \geq n_0$.

7. Theorem 2.2.1 If $\{a_n\}$ and $\{b_n\}$ are convergent with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

   (a) $\lim_{n\to\infty} (a_n + b_n) = a + b$
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|(a_n + b_n) - (a + b)| < \epsilon$.
   $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$.

   (b) $\lim_{n\to\infty} (a_n b_n) = ab$
   $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |b||a_n - a|$

   (c) Furthermore, if $a \neq 0$ and $a_n \neq 0 \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$
   First show that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.
   $|\frac{1}{a_n} - \frac{1}{a}| = |\frac{a - a_n}{a a_n}| = \frac{|a_n - a|}{|a_n||a|} = |a_n - a| \frac{1}{|a|} \frac{1}{|a_n|}$.
   Note that $|a| = |a - a_n + a_n| \leq |a_n - a| + |a_n|$
   Then apply (b).

8. Corollary 2.2.2: If $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = a$ and $c \in \mathbb{R}$, then

   (a) $\lim_{n\to\infty} (a_n + c) = a + c$
   $\lim_{n\to\infty} (a_n + c) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} c$ (from 2.2.1(a), since $\{a_n\}$ and $\{c\}$ converge) = a + c (since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} c = c$).

   (b) $\lim_{n\to\infty} (a_n c) = ac$
   $\lim_{n\to\infty} (a_n c) = \lim_{n\to\infty} a_n \lim_{n\to\infty} c$ (from 2.2.1(b), since $\{a_n\}$ and $\{c\}$ converge) $= ac$ (since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} c = c$).

9. Theorem 2.2.3: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|(a_n b_n) - 0| < \epsilon$. First note that $|(a_n b_n) - 0| = |a_n b_n| = |a_n||b_n|$. Now $\{b_n\}$ is bounded $\implies \exists\ M > 0$ such that $|b_n| \leq M \forall n \in \mathbb{N}$.

10. Theorem 2.2.4: Suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences for which $\exists n_0 \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n \forall n \geq n_0$. If $\{a_n\}$ and $\{c_n\}$ converge to $L$, then $\{b_n\}$ converges to $L$.
    Let $\epsilon > 0$. Then $\exists n_1 \in \mathbb{N}$ such that $a - \epsilon < a_n \forall n \geq n_1$ and $\exists n_2 \in \mathbb{N}$ such that $c_n < a + \epsilon \forall n \geq n_2$. Choose $n_3 =$ Max $\{n_0, n_1, n_2\}$.

11. If $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = a$ and $a_n \geq 0\ \forall n \in \mathbb{N}$. Then $a \geq 0$.
    Let $\epsilon > 0$. $\exists\ n_0 \in \mathbb{N}$ such that $a_{n_0} - a < \epsilon \implies a > -\epsilon$.

12. If $\{a_n\}$ and $\{b_n\}$ are convergent with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, and $a_n \leq b_n\ \forall n \in \mathbb{N}$. Then $a \leq b$.
    Now $b_n - a_n$ converges to $b - a$ and $b_n - a_n \geq 0$. The result follows from the previous problem.

13. If the sequence $\{a_n\}$ converges to $c$ and $a \leq a_n \leq b\ \forall n \in \mathbb{N}$, then $a \leq c \leq b$.
    $a_n - a \geq 0\ \forall n \in \mathbb{N} \implies$ (from 2.2.4) $c - a \geq 0$ and $b - b_n \geq 0\ \forall n \in \mathbb{N} \implies b - c \geq 0$.

14. Prove that if the sequence $\{a_n\}$ converges to $a$ and $a < c$, then there exists an $n_0$ such that $a_n < c$ for all $n \geq n_0$. $a_n$ converges to $a \implies$ *exists* $n_0 \in \mathbb{N}$ such if $n \geq n_0$, then $|a_n - a| < \epsilon\ \forall \epsilon > 0$. So if $n \geq n_0$, then $a - \epsilon < a_n < a + \epsilon\ \forall \epsilon > 0$. Hence if $n \geq n_0$, then $a_n < a + \epsilon\ \forall \epsilon > 0$. Let $\epsilon = c - a$ ($a < c \implies c - a > 0$). Therefore, $a_n < a + c - a\ \forall n \geq n_0$ or $a_n < c\ \forall n \geq n_0$.