Math 553 Homework #5 (SOLUTIONS) DUE: November 10, 2009

Exercises 2.4 (page 72): Problems:

7. Determine the limit points and the isolated points of each of the following sets.

d. $\mathbb{N}$

Let $p \in \mathbb{R}$. Show that p is not a limit point of $\mathbb{N}$

Let $p \in \mathbb{N}$. Then $N_1(p) \cap \mathbb{N} \setminus \{p\} = (p-1, p+1) \cap \mathbb{N} \setminus \{p\} = \emptyset$. Thus no integer is a limit point of $\mathbb{N}$.

Let $p \in \mathbb{R} \setminus \mathbb{N}$ and $p < 1$. Then $1 - p > 0$ and $N_{1-p}(p) \cap \mathbb{N} \setminus \{p\} = (2p-1, 1) \cap \mathbb{N} \setminus \{p\} = \emptyset$. Thus p is not a limit point of $\mathbb{N}$.

Let $p \in \mathbb{R} \setminus \mathbb{N}$, and $p > 1$. Then $\exists n \in \mathbb{N}$ such that $n < p < n+1$. Let $\epsilon =$Min $\{p - n, (n+1) - p\}$. Then $\epsilon > 0$. Now $\epsilon \leq p - n$ and $\epsilon \leq (n+1) - p$. Thus $n \leq p - \epsilon$ and $p + \epsilon \leq (n+1)$. So $n \leq p - \epsilon < p + \epsilon \leq (n+1)$ and $N_\epsilon(p) = (p - \epsilon, p + \epsilon) \subseteq (n, n+1)$. Thus $N_\epsilon(p) \setminus \{p\} \cap \mathbb{N} = \emptyset$ and p is not a limit point of $\mathbb{N}$. Hence no real number is a limit point of $\mathbb{N}$. Thus every point of $\mathbb{N}$ is an isolated point of $\mathbb{N}$.

Exercises 3.1 (page 100): Problems:

1. Prove Theorem 3.1.5 : Every open interval in $\mathbb{R}$ is an open subset of $\mathbb{R}$.

Open Intervals are: (a, b), (a, ∞), $(-\infty, a)$, and $(-\infty, \infty)$.

$(-\infty, \infty)$ is open: Let $p \in (-\infty, \infty)$. Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) \subseteq (-\infty, \infty)\ \forall \epsilon > 0$.

Prove (a, b) is open: Let $p \in (a, b)$. Choose $\epsilon =$ Min $\{p - a, b - p\}$. Then $\epsilon \leq p - a$ and $\epsilon \leq b - p$ Thus $a \leq p - \epsilon$ and $p + \epsilon \leq b$. So $a \leq p - \epsilon < p < p + \epsilon \leq b$. Thus $(p - \epsilon, p + \epsilon) \subseteq (a, b)$.

Prove (a, ∞) is open . Let $p \in (a, \infty)$. Find an $\epsilon > 0$ such that $N_\epsilon(p) \subset (a, \infty)$. Choose $\epsilon = p - a$. Then $\epsilon > 0$ (since $p \in (a, \infty) \implies p > a \implies p - a > 0$). Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) = (p - (p - a), p + (p - a)) = (a, 2p - a) \subset (a, \infty)$. (Note that any ϵ such that $0 < \epsilon < p - a$ will also work). So (a, ∞) is open.

Prove that $(-\infty, a)$ is open: Let $p \in (-\infty, a)$. Find an $\epsilon > 0$ such that $N_\epsilon(p) \subset (-\infty, a)$. Choose $\epsilon = a - p$. Then $\epsilon > 0$ (since $p \in (-\infty, a) \implies p < a \implies a - p > 0$). Then $N_\epsilon(p) = (p - \epsilon, p + \epsilon) = (p - (a - p), p + (a - p)) = (2p - a, a) \subset (-\infty, a)$. (Note that any ϵ such that $0 < \epsilon < a - p$ will also work). So $(-\infty, a)$ is open.

2. Show that every finite subset of $\mathbb{R}$ is closed.

Let E be a finite subset of $\mathbb{R}$ and let E' be the set of limit points of E. Then, by the fact that a finite subset of $\mathbb{R}$ has no limit points, $E' = \emptyset$. Thus $E' = \emptyset \subseteq E$. Hence E contains all of its limit points. Therefore, E is closed.

3. Show that the intervals $(-\infty, a]$ and $[a, \infty]$ are closed subsets of $\mathbb{R}$. $[a, \infty)^c = (-\infty, a)$, an open set (as shown above). Thus the complement of $[a, \infty)$ is open. Hence $[a, \infty)$ is closed (by definition of a closed set).

$(-\infty, a]^c = (a, \infty)$, an open set (as shown above). Thus the complement of $(-\infty, a]$ is open. Hence $(-\infty, a]$ is closed (by definition of a closed set).

5 a. Let F be a closed subset of $\mathbb{R}$ and let $\{p_n\}$ be a sequence in F which converges to $p \in \mathbb{R}$. Prove that $p \in F$.

Assume $p \in F^c$ Prove: a contradiction. We will show that p is a limit of F. Let $\epsilon > 0$. Show that $N_\epsilon(p)$ contains a point of F other that p. Now $p_n \to p \implies \exists n_0 \in \mathbb{N}$ such that $|p_n - p| < \epsilon \forall n \geq n_0$ $\iff p - \epsilon < p_n < p + \epsilon \forall n \geq n_0$. Thus $p_n \in N_\epsilon(p) = (p - \epsilon, p + \epsilon) \forall n \geq n_0$. In particular, $p_{n_0} \in N_\epsilon(p) = (p - \epsilon, p + \epsilon)$. Now $p_{n_0} \neq p \in F$ since $p \in F^c$. Thus $N_\epsilon(p)$ contains a point of F other that p. Hence p is a limit point of F and so, since F is closed, $p \in F$, a contradiction.

6. a. Prove Theorem 3.1.6 (a) For any collection $\{O_\alpha : \alpha \in I\}$ of open subsets of $\mathbb{R}$, $\cup_{\alpha \in I} O_\alpha$ is open. Let $p \in \cup_{\alpha \in I} O_\alpha$. Then $p \in O_\alpha$ for some $\alpha \in I$. Since O_α is open, and $p \in O_\alpha$ $\exists$ an $\epsilon > 0$ such that $N_\epsilon(p) \subset O_\alpha$ (def. of an open set). Then $N_\epsilon(p) \subset O_\alpha \subset \cup_{\alpha \in I} O_\alpha$. Hence $\cup_{\alpha \in I} O_\alpha$ contains a ϵ-nhd of each of its points and is, therefore, open. b. Give an example of an infinite collection $\{F_n\}_{n=1}^\infty$ of closed subsets of $\mathbb{R}$ such that $\cup_{n=1}^\infty F_n$ is not closed. Consider the infinite family family $\{[\frac{1}{n}, 1] : n \in \mathbb{N}\}$ of closed subsets of $\mathbb{R}$. Then, we will prove that $\cup_{n=1}^\infty [\frac{1}{n}, 1] = (0, 1]$, which is not closed.

Since $0 < \frac{1}{n} \forall n \in \mathbb{N}$, $\cup_{n=1}^\infty [\frac{1}{n}, 1] \subseteq (0, 1]$. Now let $x \in (0, 1]$. Then $x > 0$ and $x \leq 1$. Now $x > 0 \implies \exists n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$ (Archimedean Property). But $x \leq 1$. So $x \in [\frac{1}{n_0}, 1]$ and $[\frac{1}{n_0}, 1] \subseteq \cup_{n=1}^\infty [\frac{1}{n}, 1]$. Thus $(0, 1] \subseteq \cup_{n=1}^\infty [\frac{1}{n}, 1]$. Combining this with $\cup_{n=1}^\infty [\frac{1}{n}, 1] \subseteq (0, 1]$, we have $\cup_{n=1}^\infty [\frac{1}{n}, 1] = (0, 1]$.

10. Prove that the set of limit points of a set is closed. Let $E \subset \mathbb{R}$ and E' be the set of limit points of E. We need to prove that E' is closed. So we will show that E' contains all of its limit points ($F \subset \mathbb{R}$ is closed $\iff$ F contains all of its limit points). Let p be a limit point of E'. Prove: $p \in E'$. We

will show that p is a limit point of E (since $p \in E' \iff p$ is a limit point of E). Let $\epsilon > 0$ and prove $N_\epsilon(p) \cap E \setminus \{p\} \neq \emptyset$; that is, E contains a point of $N_\epsilon(p)$, other than p. Now $N_\epsilon(p) \cap E' \setminus \{p\} \neq \emptyset$ (since p is a limit point of E'). Thus $\exists q \in N_\epsilon(p) \cap E' \setminus \{p\} \implies q \in N_\epsilon(p)$ and $q \in E' \setminus \{p\}$. $q \in E' \setminus p \implies q$ is a limit point of E(definition of E') and $q \in N_\epsilon(p) \implies \exists$ a $\delta > 0$ such that $N_\delta(q) \subset N_\epsilon(p)$ (since $N_\epsilon(p) = (p-\epsilon, p+\epsilon)$ is open). Since q is a limit point of E, $N_\delta(q)$ has infinitely many points of E and hence a point of E, other than p. Thus $N_\epsilon(p)$, (since $N_\delta(q) \subset N_\epsilon(p)$), has a point of E, other than p.

1. Give an example of an infinite collection $\{O_n\}_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$ such that $\cap_{n=1}^{\infty} O_n$ is not open.
   Consider the infinite family $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ of open subsets of $\mathbb{R}$. Then, we will prove that $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open.
   $\{0\} \subseteq \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$, since $0 \in (-\frac{1}{n}, \frac{1}{n}) \forall n \in \mathbb{N}$. Now let $x \in \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) \forall n \in \mathbb{N}$. Let $x > 0$. By the Archimedean property, $\exists n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Thus $x \notin (-\frac{1}{n_0}, \frac{1}{n_0})$. So $x \notin \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$. Let $x < 0$. Then $-x > 0$. By the Archimedean property, $\exists n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < -x$. Thus, $x < -\frac{1}{n_1}$ and $x \notin (-\frac{1}{n_1}, \frac{1}{n_1})$. So $x \notin \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$. Hence $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) \subseteq \{0\}$. Combining with $\{0\} \subseteq \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$, we have $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

2. Let E be a subset of $\mathbb{R}$ and $p \in \mathbb{R}$. If there exists a sequence $\{p_n\}$ of points in $E \setminus \{p\}$ that converges to p, then p is a limit point of E.
   We show that p is a limit point of E. Let $\epsilon > 0$ and prove $N_\epsilon(p) \cap E \setminus \{p\} \neq \emptyset$; that is, E contains a point of $N_\epsilon(p)$, other than p.
   Since $p_n \longrightarrow p$, $\exists n_0 \in \mathbb{N}$ such that $|p_n - p| < \epsilon \ \ \forall n \geq n_0$; that is, $p_n \in N_\epsilon(p) \ \ \forall n \geq n_0$. In particular, $p_{n_0} \in N_\epsilon(p)$. Also, $p_{n_0} \in E \setminus \{p\}$, since $\{p_n\}$ is a sequence of points in $E \setminus \{p\}$. Hence $N_\epsilon(p) \cap E \setminus \{p\} \neq \emptyset$.

3. Let E be a closed and bounded set. Prove that E contains its infimum and supremum.
   In Homework # 2, we proved that
   8. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above and let $\alpha = \sup A$. If $\alpha \notin A$, prove that α is a limit point of A.
   and (Last problem) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below and let $\beta = \inf A$. If $\beta \notin A$, prove that β is a limit point of A
   Restate: Let A be a nonempty subset of $\mathbb{R}$ that is bounded and let $\alpha = \sup A$ and let $\beta = \inf A$. Then
   (1) $\alpha \in A$ or α is a limit point of A.
   (2) $\beta \in A$ or β is a limit point of A.
   Now E is a bounded set. Let $\alpha = \sup E$ and let $\beta = \inf E$. If $\alpha \in E$, we are done. Assume $\alpha \notin E$. By (1)above, α is a limit point of E. But E is closed (given) $\implies$ E contains all of its limit points. Hence $\alpha \in E$. If $\beta \in E$, we are done. Assume $\beta \notin E$. By (2)above, β is a limit point of E. But E is closed (given) $\implies$ E contains all of its limit points. Hence $\beta \in E$.

4. Suppose A is an open set and B is a closed set. Prove that $A \setminus B$ is an open set and $B \setminus A$ is a closed set.
   $A \setminus B = \{x | x \in A \text{ and } x \notin B\} = \{x | x \in A \text{ and } x \in B^c\} = A \cap B^c$. Now A is open (given) and B^c is open, since it is given that B is closed. Moreover, the intersection of a finite number of open sets is open. Hence $A \setminus B = A \cap B^c$ is open. $B \setminus A = \{x | x \in B \text{ and } x \notin A\} = \{x | x \in B$ and $x \in A^c\} = B \cap A^c$. Now B is closed (given) and A^c is closed, since it is given that A is open. Moreover, the intersection of any number of closed sets is closed. Hence $B \setminus A = B \cap A^c$ is closed.