Homework #4 (SOLUTIONS)

Math 553 DUE: November 3, 2009

`Exercises 2.4 (page 72): Problems:`

3. Find all the subsequential limits of the following sequences.
b. $\{\cos\frac{n\pi}{2}\}$
$\{\cos\frac{(4k)\pi}{2}\} = \{1\}$, $\{\cos\frac{(4k+1)\pi}{2}\} = \{0\}$, and $\{\cos\frac{(4k+2)\pi}{2}\} = \{-1\}$. Thus the set of subsequential limits is $\{0, 1, -1\}$.
f. $\{(1.5 + (-1)^n)^n\}$.
$\{(1.5 + (-1)^{2k-1})^{2k-1}\} = (\frac{1}{2})^{2k-1}$ cgs to 0
and $\{(1.5 + (-1)^{2k})^{2k}\} = (2.5)^{2k}$ dgs to ∞. Thus the set of subsequential limits is $\{0, \infty\}$.
7. Determine the limit points and the isolated points of each of the following sets.
c. $(0, 1) \cup \{2\}$
The set of limit points is $[0, 1]$ and 2 is the only isolated point of $(0, 1) \cup \{2\}$
e. $\mathbb{R} \setminus \mathbb{Q}$
Let $p \in \mathbb{R}$. Show that p is a limit point of $\mathbb{R} \setminus \mathbb{Q}$.
Let $\epsilon > 0$. Then since $p < p+\epsilon$, $\exists r \in \mathbb{R}\setminus\mathbb{Q}$ such that $p < r < p+\epsilon$. Thus $N_\epsilon(p) = (p-\epsilon, p+\epsilon)$ contains a point of $\mathbb{R} \setminus \mathbb{Q}$ other than p. $\implies$ p is a limit point of $\mathbb{R} \setminus \mathbb{Q}$. Hence the set of limit points of $\mathbb{R} \setminus \mathbb{Q}$ is $\mathbb{R}$. Thus $\mathbb{R} \setminus \mathbb{Q}$ has no isolated points.
f. $\mathbb{Q} \cap (0, 1)$
The set of limit points is $[0, 1]$ and $\mathbb{Q} \cap (0, 1)$ has no isolated points.
8. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above and let $\alpha = \sup A$. If $\alpha \notin A$, prove that α is a limit point of A.
Let $\epsilon > 0$. We will show that $N_\epsilon(\alpha)\cap A\setminus\{\alpha\} \neq \emptyset$; that is, $N_\epsilon(\alpha)\cap A \neq \emptyset$ (since $\alpha \notin A \implies A\setminus\{\alpha\} = A$). Since $\alpha-\epsilon < \alpha$, $\alpha-\epsilon$ is not an upper bound of A (since $\alpha = \sup A$). $\implies \exists\, a \in A$ such that $\alpha-\epsilon < a \leq \alpha$. Thus $(\alpha-\epsilon, \alpha+\epsilon)\cap A \neq \emptyset \implies N_\epsilon(\alpha)\cap A \neq \emptyset$, since $a \in N_\epsilon(\alpha)$ and $a \in A$. Hence α is a limit point of A.

`Exercises 2.6 (page 85): Problems:`

1. If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $\mathbb{R}$, prove (without using Theorem 2.6.4) that $\{a_n + b_n\}$ and $\{a_n b_n\}$ are Cauchy.
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then $|(a_n + b_n) - (a_m + b_m)| < \epsilon$.
$\{a_n\}$ is Cauchy $\implies \exists n_1 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}\forall m, n \geq n_1$ and $\{b_n\}$ is Cauchy $\implies \exists n_2 \in \mathbb{N}$ such that $|b_n - b_m| < \frac{\epsilon}{2}\forall m, n \geq n_2$. Choose $n_0 = \text{Max }\{n_1, n_2\}$. If $m, n \geq n_0$ then $|(a_n + b_n) - (a_m + b_m)| = |a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Next, $\{a_n b_n\}$ is Cauchy.
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then $|(a_n b_n) - (a_m b_m)| < \epsilon$.
$\{a_n\}$ and $\{b_n\}$ are Cauchy sequences $\implies$ both are bounded $\implies \exists\ M_1 > 0$ and $M_2 > 0$ such that $|a_n| < M_2 \forall n \in \mathbb{N}$ and $|b_n| < M_1 \forall n \in \mathbb{N}$. Also $\{a_n\}$ is Cauchy $\implies \exists n_1 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2M_1}\forall m, n \geq n_1$ and $\{b_n\}$ is Cauchy $\implies \exists n_2 \in \mathbb{N}$ such that $|b_n - b_m| < \frac{\epsilon}{2M_2}\forall m, n \geq n_2$. Choose $n_0 = \text{Max }\{n_1, n_2\}$. If $m, n \geq n_0$ then $|(a_n b_n)-(a_m b_m)| = |(a_n b_n)-(a_n b_m)+(a_n b_m)-(a_m b_m)| = |a_n(b_n - b_m) + b_m(a_n - a_m)| \leq |a_n||b_m - b_n| + |b_m||a_m - a_n| < M_2(\frac{\epsilon}{2M_2}) + M_1(\frac{\epsilon}{2M_1}) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

1. Determine three subsequences, of the sequence $\{\sin\frac{\pi\sqrt{n}}{2}\}$, each converging to a different limit.
   $\{\sin\frac{(4k-3)\pi}{2}\} = \{1\}$ converges to 1.
   $\{\sin\frac{(4k-1)\pi}{2}\} = \{-1\}$ converges to -1, and
   $\{\sin\frac{(2k)\pi}{2}\} = \{\sin k\pi\} = \{0\}$ converges to 0.

2. Use only the definition to show that the following sequences are or are not Cauchy.
   (a) $\{\frac{2^n-1}{2^n}\}$
   We prove that $\{\frac{2^n-1}{2^n}\}$ is Cauchy.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then $|\frac{2^n-1}{2^n} - \frac{2^m-1}{2^m}| < \epsilon$. Now $|\frac{2^n-1}{2^n} - \frac{2^m-1}{2^m}| =$

$|\frac{2^m(2^n-1)-2^n(2^m-1}{2^m 2^n}| = \frac{|2^m-2^n|}{2^m 2^n} = \frac{2^m-2^n}{2^m 2^n}$ (assume $m > n$) $< \frac{2^m}{2^m 2^n} = \frac{1}{2^n} < \frac{1}{n} < \epsilon$.
$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$. Choose $n_0 > \frac{1}{\epsilon}$. Then if $m > n \geq n_0$ then $|\frac{2^n-1}{2^n} - \frac{2^m-1}{2^m}| < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$.

(b) $\{\frac{2n^2+1}{n^2}\}$
We prove that $\{\frac{2n^2+1}{n^2}\}$ is Cauchy.
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then $|\frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2}| < \epsilon$. Now $|\frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2}| = |\frac{m^2(2n^2+1)-n^2(2m^2+1}{m^2n^2}| = \frac{|m^2-n^2|}{m^2n^2} = \frac{m^2-n^2}{m^2n^2}$ (assume $m > n$) $< \frac{m^2}{m^2n^2} = \frac{1}{n^2} < \frac{1}{n} < \epsilon$.
$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$. Choose $n_0 > \frac{1}{\epsilon}$. Then if $m > n \geq n_0$ then $|\frac{2n^2+1}{n^2} - \frac{2m^2+1}{m^2}| < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$.

3. Use the definition to prove that the sequence $\{\frac{1}{n^2}\}$ is Cauchy.
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then $|\frac{1}{n^2} - \frac{1}{m^2}| < \epsilon$. Now $|\frac{m^2-n^2}{m^2n^2}| = \frac{m^2-n^2}{m^2n^2}$ (assume $m > n$) $< \frac{m^2}{m^2n^2} = \frac{1}{n^2} < \frac{1}{n} < \epsilon$.
$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$. Choose $n_0 > \frac{1}{\epsilon}$. Then if $m > n \geq n_0$ then $|\frac{1}{m^2} - \frac{1}{n^2}| < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$.

4. Use the definition to prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence.
Prove: $\exists \epsilon > 0\ \forall n_0 \in \mathbb{N}$ such that $\exists m \geq n_0$ and $\exists n \geq n_0$ and $|(-1)^m - (-1)^n| \geq \epsilon$.
Let $n_0 \in \mathbb{N}$. Choose $\epsilon = 2$. Then $n = n_0 \geq n_0$ and $m = n_0 + 1 \geq n_0$, and $|(-1)^n - (-1)^m| = |(-1)^{n_0} - (-1)^{n_0+1}| = 2 \geq \epsilon$.

5. Use the definition to prove that the sequence $\{n\}$ is not a Cauchy sequence.
Prove: $\exists \epsilon > 0\ \forall n_0 \in \mathbb{N}$ such that $\exists m \geq n_0$ and $\exists n \geq n_0$ and $|n - m| \geq \epsilon$.
Let $n_0 \in \mathbb{N}$. Choose $\epsilon = 1$. Then $n = n_0 \geq n_0$ and $m = n_0 + 1 \geq n_0$, and $|m - n| = |n_0 - (n_0 + 1)| = 1 \geq \epsilon$.

6. If $\{a_n\}$ has two subsequences that converge to different limits, then prove that $\{a_n\}$ diverges.
Theorem: If $\{a_n\}$ converges, then every subsequence of $\{a_n\}$ converges to $\underset{n\to\infty}{lim} a_n$.
An equivalent statement is:
If every subsequence of $\{a_n\}$ does not converge to $\underset{n\to\infty}{lim} a_n$, then $\{a_n\}$ diverges.
Now $\{a_n\}$ has two subsequences that converge to different limits $\implies$ every subsequence of $\{a_n\}$ does converges to $\underset{n\to\infty}{lim} a_n$. Hence, by the Theorem above, the sequence $\{a_n\}$ diverges.

7. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below and let $\beta = \inf A$. If $\beta \notin A$, prove that β is a limit point of A
Let $\epsilon > 0$. We will show that $N_\epsilon(\beta) \cap A \setminus \{\beta\} \neq \emptyset$; that is, $N_\epsilon(\beta) \cap A \neq \emptyset$ (since $\beta \notin A \implies A \setminus \{\beta\} = A$). Then, since $\beta < \beta + \epsilon$, $\beta + \epsilon$ is not a lower bound of A (since $\beta = \inf A$). $\implies \exists\ a \in A$ such that $\beta \leq a < \beta + \epsilon$. Thus $(\beta - \epsilon, \beta + \epsilon) \cap A \neq \emptyset \implies N_\epsilon(\beta) \cap A \neq \emptyset$, since $a \in N_\epsilon(\beta)$ and $a \in A$. Hence β is a limit point of A.