Homework #3 (SOLUTIONS)

Math 553 DUE: October 27, 2009

1. If the sequence $\{a_n\}$ is decreasing and bounded below then prove that $\{a_n\}$ converges.
   $\{a_n\}$ is bounded below $\implies A = \{a_n : n \in \mathbb{N}\}$ is bounded below and nonempty. Let $\beta = \inf A$. Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n - \beta| < \epsilon$. Now $\beta < \beta + \epsilon$. So $\beta + \epsilon$ is not a lower bound of A. Then $\exists\ a_{n_0} \in A$ such that $a_{n_0} < \beta + \epsilon$. But $\{a_n\}$ is decreasing $\implies a_n \leq a_{n_0} \forall n \geq n_0$. Thus, $a_n \leq a_{n_0} < \beta + \epsilon\ \forall n \geq n_0$.
   Hence, $\beta \leq a_n < \beta + \epsilon\ \forall n \geq n_0$ or $\beta - \epsilon < a_n < \beta + \epsilon\ \forall n \geq n_0$

2. Use the definition to prove that the following are true.

   (a) $\underset{n\to\infty}{lim} \sqrt{\frac{n^4+1}{25}} = +\infty$
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\sqrt{\frac{n^4+1}{25}} > M$.
   Now $\sqrt{\frac{n^4+1}{25}} = \frac{\sqrt{n^4+1}}{5} > \frac{\sqrt{n^4}}{5}$(since $n^4 + 1 > n^4$) $= \frac{n^2}{5} \geq \frac{n}{5}$. Now $\frac{n}{5} > M \implies n > 4M$. Choose $n_0 > 4M$.

   (b) $\underset{n\to\infty}{lim} \frac{n^3+11}{n^2+65} = +\infty$
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\frac{n^3+11}{n^2+65} > M$.
   Now $\frac{n^3+11}{n^2+65} > \frac{n^3}{n^2+65}$ (since $n^3 + 11 > n^3$) $\geq \frac{n^3}{n^2+n^2}$ (assume $n \geq 9$. Then $65 \leq n^2 \implies 65 + n^2 \leq n^2 + n^2 \implies \frac{1}{65+n^2} \geq \frac{1}{n^2+n^2}$) $= \frac{n^3}{2n^2} = \frac{n}{2}$. Then $\frac{n}{2} > M \implies n > 2M$. Choose $n_0 >$Max $\{9, 2M\}$.

   (c) $\underset{n\to\infty}{lim} (n^5 - n^6) = -\infty$
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $(n^5 - n^6) < -M$.
   Now $(n^5 - n^6) = -n^5(n-1) < -n^5$ (assume $n \geq 2$. $n \geq 2 \implies (n-1) \geq 1 \implies -n^5(n-1) \leq -n^5$)) $\leq -n < -M$. Thus $n > M$. Choose $n_0 >$Max $\{2, M\}$.

   (d) $\underset{n\to\infty}{lim} \frac{n^3-1}{n^2} = \infty$
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\frac{n^3-1}{n^2} > M$.
   Now $\frac{n^3-1}{n^2} \geq \frac{n^3-n^2}{n^2}$ (since $n \geq 1 \implies n^2 \geq 1 \implies -n^2 \leq -1 \implies n^3 - n^2 \leq n^3 - 1$) $= \frac{n^2(n-1)}{n^2} = n - 1$. Now $n - 1 > M \implies n > M + 1$. Choose $n_0 > M + 1$.

   (e) $\underset{n\to\infty}{lim} (n^2 + (-1)^n n) = \infty$
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $(n^2 + (-1)^n n) > M$.
   Now $(n^2 + (-1)^n n) \geq n^2 - n$ (since $n^2 + (-1)^n n = n^2 + n$, n even and $n^2 + (-1)^n n = n^2 - n$, n odd) $= n(n-1) \geq n$ (assume $n \geq 2$. Then $n - 1 \geq 1 \implies n(n-1) \geq n$). Choose $n_0 >$Max $\{2, M\}$.

3. If the sequence $\{a_n\}$ is increasing and not bounded above, then prove that $\{a_n\}$ diverges to ∞.
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $a_n > M$.
   $\{a_n\}$ is unbounded $\implies A = \{a_n : n \in \mathbb{N}\}$ is unbounded above. Thus, $\exists$ an $n_0 \in \mathbb{N}$ such that $a_{n_0} \in A$ and $a_{n_0} > M$. Now $\{a_n\}$ is increasing $\implies a_n \geq a_{n_0}\ \forall n \geq n_0$. Hence, if $n \geq n_0$ then $a_n > M$. So $a_n \longrightarrow \infty$.

4. If $a_n \longrightarrow -\infty$ and $\{b_n\}$ converges, then prove that $\{a_n + b_n\}$ diverges to $-\infty$.
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\{a_n + b_n\} < -M$.
   Assume $\{b_n\}$ converges to b. Then $\exists n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then $|b_n - b| < \frac{M}{2}$; that is, if $n \geq n_1$, then $b_n - b < \frac{M}{2}$ or if $n \geq n_1$, then $b_n < b + \frac{M}{2}$. Now $a_n \longrightarrow -\infty \implies \exists$ an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $a_n < -\frac{3M}{2} - b$. Choose $n_0 =$ Max $\{n_1, n_2\}$. Then, if $n \geq n_0$, then (since $n \geq n_1$ and $n \geq n_2$) $a_n + b_n < (-\frac{3M}{2} - b) + (b + \frac{M}{2}) = -M$.

5. If $a_n > 0$ for all n and $\{a_n\}$ converges to 0, then prove that $\{\frac{1}{a_n}\}$ diverges to ∞.
   Let $M > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\frac{1}{a_n} > M$.
   Now $\{a_n\}$ converges to 0 $\implies$ $\exists$ an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n - 0| < \frac{1}{M}$. Now $|a_n - 0| = |a_n| = a_n$ for all $n \in \mathbb{N}$ (since $a_n > 0 \forall n \in \mathbb{N}$). So, if $n \geq n_0$, then $|a_n - 0| = a_n < \frac{1}{M}$.
   Hence, if $n \geq n_0$, then $\frac{1}{a_n} > M$.

`Exercises 2.3 (page 66): Problems:`

7. For each of the following, prove that the sequence $\{a_n\}$ converges and find the limit.
a. $a_{n+1} = \frac{1}{6}(2a_n + 5)$, $a_1 = 2$.

1. Show $\{a_n : n \in \mathbb{N}\}$ is bounded below.
   We will show, by induction, that $a_n > 1$ $\forall n \geq 1$.
   Basis of induction: Since $a_1 = 2 > 1$, the statement holds for $n = 1$.
   Induction: Let $n \geq 1$ be fixed and that $a_n > 1$.
   We now prove that $a_{n+1} > 1$.
   $a_{n+1} = \frac{1}{6}(2a_n + 5) > \frac{1}{6}(2(1) + 5)$ (since $a_n > 1$) $= \frac{7}{6} > 1$.

2. We now prove, by induction, that $\{a_n\}$ is decreasing; that is,
   $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$.
   Basis of induction: Since $a_1 = 2 \geq \frac{3}{2} = \frac{1}{6}(2(2) + 5) = \frac{1}{6}(2a_1 + 5) = a_2$, the statement holds for $n = 1$.
   Induction: Let $n \geq 1$ be fixed and that $a_n \geq a_{n+1}$.
   We now prove that $a_{n+1} \geq a_{n+2}$.
   But $a_{n+1} = \frac{1}{6}(2a_n + 5) \geq \frac{1}{6}(2a_{n+1} + 5)$ (by induction) $= a_{n+2}$ (by definition of a_n).
   We have proved that $\{a_n\}$ is bounded below and decreasing. Therefore, $\{a_n\}$ converges, say to a.
   Now $a = \underset{n\to\infty}{lim} a_n = \underset{n\to\infty}{lim} a_{n+1} = \underset{n\to\infty}{lim} \frac{1}{6}(2a_n + 5) = \frac{2}{6} \underset{n\to\infty}{lim} a_n + \frac{5}{6} = \frac{1}{3}a + \frac{5}{6}$.
   So, $a = \frac{1}{3}a + \frac{5}{6}$. Hence $\frac{2}{3}a = \frac{5}{6}$ or $a = \frac{5}{4}$.