Math 553 Homework #2($SOLUTIONS$) DUE: October 8, 2009

1. Prove using the $\epsilon - n_0$ definition that the following sequences converge.

   (a) $\{\frac{6n-2}{5n-7}\}$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\frac{6n-2}{5n-7} - \frac{6}{5}| < \epsilon \; \forall n \geq n_0$. $|\frac{6n-2}{5n-7} - \frac{6}{5}| = |\frac{5(6n-2)-6(5n-7)}{5(5n-7)}| = |\frac{32}{5(5n-7)}| \leq |\frac{7}{(5n-7)}|$(since $\frac{32}{5} \leq 7$) $\leq \frac{7}{5n^2-n}$ (assume $n \geq 7 \implies 5n^2 - n \leq 5n - 7$) $= \frac{7}{4n} < \frac{2}{n}$ (since $\frac{7}{4} < 2$) $< \epsilon \implies \frac{2}{\epsilon} < n$. Choose $n_0 >$ Maximum $\{7, \frac{2}{\epsilon}\}$.

   (b) $\{\frac{80}{\sqrt{5n}}\}$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\frac{80}{\sqrt{5n}-0}| < \epsilon \; \forall n \geq n_0$.
   $|\frac{80}{\sqrt{5n}-0}| = \frac{80}{\sqrt{5n}} < \epsilon \implies \frac{80}{\epsilon} < \sqrt{5n}$. So $5n > \frac{80^2}{\epsilon^2} = \frac{6400}{\epsilon^2} \implies n > \frac{1280}{\epsilon^2}$. Choose $n_0 > \frac{1280}{\epsilon^2}$.

   (c) $\{\frac{\cos n}{n^2+17}\}$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\frac{\cos n}{n^2+17} - 0| < \epsilon \; \forall n \geq n_0$.
   $|\frac{\cos n}{n^2+17} - 0| = \frac{|\cos n|}{n^2+17} \leq \frac{1}{n^2+17}$ (since $|\cos n| \leq 1 \forall n$) $\leq \frac{1}{n^2} \leq \frac{1}{n}$. Now $\frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}$. Choose $n_0 > \frac{1}{\epsilon}$.

2. Suppose that $x_n \longrightarrow a$ and $c \in \mathbb{R}$. Prove, using the $\epsilon - n_0$ definition, that $cx_n \longrightarrow ca$:
   Let $\epsilon > 0$. Find an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|cx_n - ca| < \epsilon$.
   Case (1): Let $c = 0$. Then $|cx_n - ca| = |0x_n - 0a| = 0 < \epsilon \; \forall n \geq 1$. Case(2): Let $c \neq 0$. $x_n \longrightarrow a \implies \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|x_n - a| < \frac{\epsilon}{|c|}$. Then, if $n \geq n_0$ then $|cx_n - ca| = |c(x_n - a)| = |c||x_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon$.

3. Suppose that $x_n \longrightarrow a$ and $y_n \longrightarrow b$. Prove, using the $\epsilon - n_0$ definition, that $x_n - y_n \longrightarrow a - b$:
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|(x_n - y_n) - (a - b)| < \epsilon$.
   $x_n \longrightarrow a \implies \exists n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $|x_n - a| < \frac{\epsilon}{2}$ and $y_n \longrightarrow b \implies \exists n_2 \in \mathbb{N}$ such that if $n \geq n_2$ then $|y_n - b| < \frac{\epsilon}{2}$. Choose $n_0 =$ Max $\{n_1, n_2\}$. Then, if $n \geq n_0$ then $|(x_n - y_n) - (a - b)| = |(x_n - a) + (b - y_n)| \leq |x_n - a| + |b - y_n| = |x_n - a| + |y_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

4. Prove that if $x_n \longrightarrow a$ and $|y_n - x_n| \leq \frac{1}{n}$ for all n, then $y_n \longrightarrow a$.
   $x_n \longrightarrow a \implies \exists n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $|x_n - a| < \frac{\epsilon}{2}$ and, by Archimedean Property, $\exists$ an $n_2 \in \mathbb{N}$ such that $\frac{1}{n_2} < \frac{\epsilon}{2}$. So, if $n \geq n_2$ then $\frac{1}{n} \leq \frac{1}{n_2} < \frac{\epsilon}{2}$. Choose $n_0 =$ Max $\{n_1, n_2\}$. Then, if $n \geq n_0$ then $|y_n - a| = |y_n - x_n + x_n - a| \leq |y_n - x_n| + |x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

5. (a) Prove that if $x_n \longrightarrow 0$ and $x_n \geq 0$ for all n, then $\sqrt{x_n} \longrightarrow 0$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\sqrt{x_n} - 0| < \epsilon \; \forall n \geq n_0$.
   Notice that $|\sqrt{x_n} - 0| = |\sqrt{x_n}|$. $x_n \longrightarrow 0 \implies \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|x_n - 0| = x_n$ (since $x_n \geq 0$) $< \epsilon^2$. Hence (since $x_n < \epsilon^2 \implies \sqrt{x_n} < \epsilon$) if $n \geq n_0$ then $|\sqrt{x_n} - 0| = |\sqrt{x_n}| = \sqrt{x_n} < \epsilon$.

   (b) Prove that if $x_n \longrightarrow a > 0$ and $x_n \geq 0$ for all n, then $\sqrt{x_n} \longrightarrow \sqrt{a}$.
   Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\sqrt{x_n} - \sqrt{a}| < \epsilon \; \forall n \geq n_0$. $|\sqrt{x_n} - \sqrt{a}| = |\frac{(\sqrt{x_n}-\sqrt{a})(\sqrt{x_n}+\sqrt{a})}{\sqrt{x_n}+\sqrt{a}}| = |\frac{x_n - a}{\sqrt{x_n}+\sqrt{a}}| = \frac{|x_n - a|}{\sqrt{x_n}+\sqrt{a}} \leq \frac{|x_n - a|}{\sqrt{a}}$ (since $x_n \geq 0$). Now $x_n \longrightarrow a \implies \exists n_0 \in \mathbb{N}$ such that $|x_n - a| < \epsilon\sqrt{a}$. Hence if $n \geq n_0$ then $|\sqrt{x_n} - \sqrt{a}| \leq \frac{|x_n - a|}{\sqrt{a}} < \epsilon\sqrt{a}\frac{1}{\sqrt{a}} = \epsilon$.

6. Prove that if $x_n \longrightarrow a$ and $a < c$, then there exists an $n_0$ such that $x_n < c$ for all $n \geq n_0$.
   Let $\epsilon = c - a > 0$ (since $a < c$). $a_n$ converges to $a \implies \exists \; n_0 \in \mathbb{N}$ such if $n \geq n_0$, then $|a_n - a| < \epsilon$. So if $n \geq n_0$, then $a - \epsilon < a_n < a + \epsilon$. Thus, if $n \geq n_0$, then $a_n < a + \epsilon = a + (c - a) = c$.

Exercises 2.1 (page 52):
8. For each of the following sequences, prove, using an $\epsilon - n_0$ argument that the sequence converges to the given limit $a$; that is, given $\epsilon > 0$, determine $n_0$ such that $|a_n - a| < \epsilon \; \forall n \geq n_0$.
b. $\{\frac{2n+5}{6n-3}\}$, $a = \frac{1}{3}$. Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\frac{2n+5}{6n-3} - \frac{1}{3}| < \epsilon \; \forall n \geq n_0$. $|\frac{2n+5}{6n-3} - \frac{1}{3}| = |\frac{3(2n+5)-(6n-3)}{3(6n-3)}| = |\frac{18}{3(6n-3)}| = |\frac{6}{(6n-3)}| = \frac{6}{(6n-3)} \leq \frac{6}{6n-n}$ (assume $n \geq 3$) $= \frac{6}{5n} < \epsilon \implies \frac{6}{5\epsilon} < n$. Choose $n_0 >$ Maximum $\{3, \frac{6}{5\epsilon}\}$.

d. $\{1 - \frac{(-1)^n}{n}\}$, $a = 1$. Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|1 - \frac{(-1)^n}{n} - 1| < \epsilon \; \forall n \geq n_0$. $|1 - \frac{(-1)^n}{n} - 1| = \frac{|(-1)^n|}{n}$ (since n $\geq 1$) $= \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$. Choose $n_0 > \frac{1}{\epsilon}$.

e. $\{\frac{(-1)^n n}{n^2+1}\}$, $a = 0$. Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|\frac{(-1)^n n}{n^2+1} - 0| < \epsilon \; \forall n \geq n_0$. $|\frac{(-1)^n n}{n^2+1} - 0| = |\frac{(-1)^n n}{n^2+1}| = \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$. Choose $n_0 > \frac{1}{\epsilon}$.

13. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} a_n = a$. Prove that $\lim_{n\to\infty} (a_n)^3 = a^3$
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $|(a_n)^3 - a^3| < \epsilon \; \forall n \geq n_0$. $|(a_n)^3 - a^3| = |(a_n - a)((a_n)^2 + a_n a + a^2)| \leq |a_n - a||(a_n)^2 + a_n a + a^2|$ (since $|xy| = |x||y| \forall x, y \in \mathbb{R}$)$\leq |a_n - a|(|(a_n)^2| + |a_n a| + |a^2|)$ (since $|x + y| \leq |x| + |y| \forall x, y \in \mathbb{R}$) $= |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2)$ So $|(a_n)^3 - a^3| \leq |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2)$. Now the sequence $\{a_n\}$ converges, so $\{a_n\}$ is bounded (by Theorem 2.1.10(b)). Thus $\exists M > 0$ such that $|a_n| \leq M$ $\forall n \in \mathbb{N}$ Also, $\{a_n\}$ converges to $a \implies \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|a_n - a| < \frac{\epsilon}{(M^2 + M|a| + |a|^2)}$. Thus if $n \geq n_0$ then $|(a_n)^3 - a^3| \leq |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2) < \frac{\epsilon}{(M^2 + M|a| + |a|^2)}(M^2 + M|a| + |a|^2) = \epsilon$.

15. Prove that if $\{a_n\}$ converges to $a$, then $\{|a_n|\}$ converges to $|a|$. Is the converse true?
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $||a_n| - |a|| < \epsilon$. First note that $||a_n| - |a|| \leq |a_n - a| \forall n \in \mathbb{N}$ (since $||x| - |y|| \leq |x - y| \forall x, y \in \mathbb{R}$). Now $\{a_n\}$ converges to $a \implies \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|a_n - a| < \epsilon$. Thus, if $n \geq n_0$ then $||a_n| - |a|| \leq |a_n - a|$ ($\forall n \in \mathbb{N}$) $< \epsilon$ (since $n \geq n_0$).

The converse, if $\{|a_n|\}$ converges to $|a|$ then $\{a_n\}$ converges to $a$, is FALSE. For example, $\{|(-1)^n|\} = \{1\}$ converges (to 1), but $\{(-1)^n\}$ diverges.

Exercises 2.2 (page 59):
1. Prove Theorem 2.2.1(a): If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} (a_n + b_n) = a + b$
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|(a_n + b_n) - (a + b)| < \epsilon$.
$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$. Now $\lim_{n\to\infty} a_n = a \implies \exists n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $|a_n - a| < \frac{\epsilon}{2}$ and $\lim_{n\to\infty} b_n = b \implies \exists n_2 \in \mathbb{N}$ such that if $n \geq n_2$ then $|b_n - b| < \frac{\epsilon}{2}$.
Choose $n_0 = Max\{n_1, n_2\}$. If $n \geq n_0$ then $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ (since $n \geq n_1$ and $n \geq n_2$) $= \epsilon$.
2. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.
a. If $\{a_n\}$ and $\{a_n + b_n\}$ both converge, prove that the sequence $\{b_n\}$ converges.
Write $b_n = (a_n + b_n) - a_n \forall n \in \mathbb{N}$. Since $\{a_n + b_n\}$ and $\{a_n\}$ converge (given), $\{b_n\} = \{(a_n + b_n) - a_n\}$ converges.

b. Suppose $b_n \neq 0 \; \forall n \in \mathbb{N}$. If $\{b_n\}$ and $\{\frac{a_n}{b_n}\}$ both converge, prove that the sequence $\{a_n\}$ also converges.
Write $a_n = (\frac{a_n}{b_n})(b_n) \forall n \in \mathbb{N}$. Since $\{\frac{a_n}{b_n}\}$ and $\{b_n\}$ are convergent, by Theorem 2.2.1 (b), $\{a_n\} = \{(\frac{a_n}{b_n})(b_n)\}$ converges.
3. Prove Theorem 2.2.3: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
Let $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|(a_n b_n) - 0| < \epsilon$. First note that $|(a_n b_n) - 0| = |a_n b_n| = |a_n||b_n|$. Now $\{b_n\}$ is bounded $\implies \exists \; M > 0$ such that $|b_n| \leq M \forall n \in \mathbb{N}$. Also, $\lim_{n\to\infty} a_n = 0 \implies \exists$ $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|a_n - 0| < \frac{\epsilon}{M}$. So if $n \geq n_0$ then $|a_n| < \frac{\epsilon}{M}$. Thus, if $n \geq n_0$ then $|(a_n b_n) - 0| = |a_n b_n| = |a_n||b_n| \leq |a_n| M$ (this works $\forall n \in \mathbb{N}$) $< \frac{\epsilon}{M} M$ (since $n \geq n_0$) $= \epsilon$.