

Exercises 2.1 (page 52):

8. For each of the following sequences, prove, using an  $\epsilon - n_0$  argument that the sequence converges to the given limit  $a$ ; that is, given  $\epsilon > 0$ , determine  $n_0$  such that  $|a_n - a| < \epsilon \forall n \geq n_0$ .

b.  $\{\frac{2n+5}{6n-3}\}$ ,  $a = \frac{1}{3}$ . Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that  $|\frac{2n+5}{6n-3} - \frac{1}{3}| < \epsilon \forall n \geq n_0$ .  $|\frac{2n+5}{6n-3} - \frac{1}{3}| = |\frac{3(2n+5)-(6n-3)}{3(6n-3)}| = |\frac{18}{3(6n-3)}| = |\frac{6}{6n-3}| = \frac{6}{6n-3} \leq \frac{6}{6n-n} = \frac{6}{5n} < \epsilon \implies \frac{6}{5\epsilon} < n$ . Choose  $n_0 > \text{Maximum}\{3, \frac{6}{5\epsilon}\}$ .

d.  $\{1 - \frac{(-1)^n}{n}\}$ ,  $a = 1$ . Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that  $|1 - \frac{(-1)^n}{n} - 1| < \epsilon \forall n \geq n_0$ .  $|1 - \frac{(-1)^n}{n} - 1| = \frac{|(-1)^n|}{n}$  (since  $n \geq 1$ )  $= \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$ . Choose  $n_0 > \frac{1}{\epsilon}$ .

e.  $\{\frac{(-1)^{n+1}}{n^2+1}\}$ ,  $a = 0$ . Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that  $|\frac{(-1)^{n+1}}{n^2+1} - 0| < \epsilon \forall n \geq n_0$ .  $|\frac{(-1)^{n+1}}{n^2+1} - 0| = \frac{|(-1)^{n+1}|}{n^2+1} = \frac{1}{n^2+1} \leq \frac{1}{n^2} = \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$ . Choose  $n_0 > \frac{1}{\epsilon}$ .

f.  $\{n(\sqrt{1 + \frac{1}{n}} - 1)\}$ ,  $a = \frac{1}{2}$ . Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that  $|n(\sqrt{1 + \frac{1}{n}} - 1) - \frac{1}{2}| < \epsilon \forall n \geq n_0$ .

$|n(\sqrt{1 + \frac{1}{n}} - 1) - \frac{1}{2}| = |n(\sqrt{1 + \frac{1}{n}} - 1) \frac{(\sqrt{1 + \frac{1}{n}} + 1)}{(\sqrt{1 + \frac{1}{n}} + 1)} - \frac{1}{2}| = |\frac{1}{\sqrt{1 + \frac{1}{n}} + 1} - \frac{1}{2}| = |\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{2}| = |\frac{\sqrt{n} - \sqrt{n+1}}{2(\sqrt{n} + \sqrt{n+1})}| = |\frac{\sqrt{n} - \sqrt{n+1}}{2(\sqrt{n} + \sqrt{n+1})} \frac{(\sqrt{n} + \sqrt{n+1})}{(\sqrt{n} + \sqrt{n+1})}| = |\frac{n - (n+1)}{2(\sqrt{n} + \sqrt{n+1})^2}| = |\frac{-1}{2(n+n+1+2\sqrt{n}\sqrt{n+1})}| = \frac{1}{2(n+n+1+2\sqrt{n}\sqrt{n+1})} = \frac{1}{2n+2n+2+2\sqrt{n}\sqrt{n+1}} < \frac{1}{2n} < \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$ . Choose  $n_0 > \frac{1}{\epsilon}$ .

13. Let  $\{a_n\}$  be a sequence in  $\mathbb{R}$  with  $\lim_{n \rightarrow \infty} a_n = a$ . Prove that  $\lim_{n \rightarrow \infty} (a_n)^3 = a^3$ .  
 Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that  $|(a_n)^3 - a^3| < \epsilon \forall n \geq n_0$ .  $|(a_n)^3 - a^3| = |(a_n - a)((a_n)^2 + a_n a + a^2)| \leq |a_n - a|(|(a_n)^2 + |a_n a| + |a^2|)$  (since  $|xy| = |x||y| \forall x, y \in \mathbb{R}$ )  $\leq |a_n - a|(|(a_n)^2| + |a_n a| + |a^2|)$  (since  $|x + y| \leq |x| + |y| \forall x, y \in \mathbb{R}$ )  $= |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2)$ . So  $|(a_n)^3 - a^3| \leq |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2)$ . Now the sequence  $\{a_n\}$  converges, so  $\{a_n\}$  is bounded (by Theorem 2.1.10(b)). Thus  $\exists M > 0$  such that  $|a_n| \leq M \forall n \in \mathbb{N}$ . Also,  $\{a_n\}$  converges to  $a \implies \exists n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $|a_n - a| < \frac{\epsilon}{(M^2 + M|a| + |a|^2)}$ . Thus if  $n \geq n_0$  then  $|(a_n)^3 - a^3| \leq |a_n - a|(|a_n|^2 + |a_n||a| + |a|^2) < \frac{\epsilon}{(M^2 + M|a| + |a|^2)}(M^2 + M|a| + |a|^2) = \epsilon$ .

15. Prove that if  $\{a_n\}$  converges to  $a$ , then  $\{|a_n|\}$  converges to  $|a|$ . Is the converse true?  
 Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $||a_n| - |a|| < \epsilon$ . First note that  $||a_n| - |a|| \leq |a_n - a| \forall n \in \mathbb{N}$  (since  $||x| - |y|| \leq |x - y| \forall x, y \in \mathbb{R}$ ). Now  $\{a_n\}$  converges to  $a \implies \exists n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $|a_n - a| < \epsilon$ . Thus, if  $n \geq n_0$  then  $||a_n| - |a|| \leq |a_n - a| < \epsilon$  ( $\forall n \in \mathbb{N}$ )  $< \epsilon$  (since  $n \geq n_0$ ).  
 The converse, if  $\{|a_n|\}$  converges to  $|a|$  then  $\{a_n\}$  converges to  $a$ , is FALSE. For example,  $\{(-1)^n\} = \{1\}$  converges (to 1), but  $\{(-1)^n\}$  diverges.

Exercises 2.2 (page 59):

1. Prove Theorem 2.2.1(a): If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences of real numbers with  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , then  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$

Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $|(a_n + b_n) - (a + b)| < \epsilon$ .  
 $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ . Now  $\lim_{n \rightarrow \infty} a_n = a \implies \exists n_1 \in \mathbb{N}$  such that if  $n \geq n_1$  then  $|a_n - a| < \frac{\epsilon}{2}$  and  $\lim_{n \rightarrow \infty} b_n = b \implies \exists n_2 \in \mathbb{N}$  such that if  $n \geq n_2$  then  $|b_n - b| < \frac{\epsilon}{2}$ . Choose  $n_0 = \text{Max}\{n_1, n_2\}$ . If  $n \geq n_0$  then  $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$  (since  $n \geq n_1$  and  $n \geq n_2$ )  $= \epsilon$ .

2. Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers.

a. If  $\{a_n\}$  and  $\{a_n + b_n\}$  both converge, prove that the sequence  $\{b_n\}$  converges.

Write  $b_n = (a_n + b_n) - a_n \forall n \in \mathbb{N}$ . Since  $\{a_n + b_n\}$  and  $\{a_n\}$  converge (given),  $\{b_n\} = \{(a_n + b_n) - a_n\}$  converges.

b. Suppose  $b_n \neq 0 \forall n \in \mathbb{N}$ . If  $\{b_n\}$  and  $\{\frac{a_n}{b_n}\}$  both converge, prove that the sequence  $\{a_n\}$  also converges.

Write  $a_n = (\frac{a_n}{b_n})(b_n) \forall n \in \mathbb{N}$ . Since  $\{\frac{a_n}{b_n}\}$  and  $\{b_n\}$  are convergent, by Theorem 2.2.1 (b),  $\{a_n\} = \{(\frac{a_n}{b_n})(b_n)\}$  converges.

3. Prove Theorem 2.2.3: Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers. If  $\{b_n\}$  is bounded and  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .

Let  $\epsilon > 0$ . Find  $n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $|(a_n b_n) - 0| < \epsilon$ . First note that  $|(a_n b_n) - 0| = |a_n b_n| = |a_n||b_n|$ . Now  $\{b_n\}$  is bounded  $\implies \exists M > 0$  such that  $|b_n| \leq M \forall n \in \mathbb{N}$ . Also,  $\lim_{n \rightarrow \infty} a_n = 0 \implies \exists n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then  $|a_n - 0| < \frac{\epsilon}{M}$ . So if  $n \geq n_0$  then  $|a_n| < \frac{\epsilon}{M}$ . Thus, if  $n \geq n_0$  then  $|(a_n b_n) - 0| = |a_n b_n| = |a_n||b_n| \leq |a_n|M$  (this works  $\forall n \in \mathbb{N}$ )  $< \frac{\epsilon}{M}M$  (since  $n \geq n_0$ )  $= \epsilon$ .