

Exercises 1.4 (page 26):

Problems:

5. Find the supremum and infimum of each of the following sets.

d. $D = \{\cos \frac{n\pi}{4} : n \in \mathbb{N}\}$

$D = \{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, 1\}$. $\sup D = 1$, $\inf D = -1$

h. $H = \{(\frac{3}{2} + (-1)^n)^n : n \in \mathbb{N}\}$

$H = \{\frac{1}{2}, (\frac{5}{2})^2, (\frac{1}{2})^3, (\frac{5}{2})^4, (\frac{1}{2})^5, (\frac{5}{2})^6, \dots\} = \{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^5}, \dots, (\frac{5}{2})^2, (\frac{5}{2})^4, \dots\}$. Thus $\sup H = \infty$ and $\inf H = 0$.

j. $J = \{x^2 : -2 < x < 2\}$

$-2 < x < 2 \implies 0 \leq x^2 < 4$. Thus $J = [0, 4)$. So $\sup J = 4$ and $\inf J = 0$.

8. Let A be a nonempty subset of \mathbb{R} . If $\alpha = \sup A$ is finite, then prove that for each $\epsilon > 0$, there is an $a \in A$ such that $\alpha - \epsilon < a \leq \alpha$.

Let $\epsilon > 0$. Then since $\alpha = \sup A$ and $\alpha - \epsilon < \alpha$, $\alpha - \epsilon$ is not an upper bound of A . So $\exists a \in A$ such that $\alpha - \epsilon < a$. Also $a \leq \alpha$ (since $\alpha = \sup A$). Hence $\exists a \in A$ such that $\alpha - \epsilon < a \leq \alpha$.

11. Use the least upper bound property of \mathbb{R} to prove that every nonempty subset of \mathbb{R} that is bounded below has an infimum.

Given: Every nonempty subset of \mathbb{R} that is bounded above has a supremum. Prove: Every nonempty subset of \mathbb{R} that is bounded below has an infimum. Proof: Let $E \neq \emptyset \subseteq \mathbb{R}$ be bounded below. Show: E has an infimum. E is bounded below $\implies \exists m \in \mathbb{R}$ such that $x \geq m \forall x \in E$. Then $-x \leq -m \forall x \in E$. Thus $F = \{-x | x \in E\}$ is bounded above by $-m$ and $F \neq \emptyset$. So F is a nonempty subset of \mathbb{R} that is bounded above. From the given, F has a supremum in \mathbb{R} , say $\alpha = \sup F$. We will prove that $-\alpha = \inf E$. Need: 1) $-\alpha$ is a lower bound of E .

Since $\alpha = \sup F$, $F = \{-x | x \in E\}$, we have $-x \leq \alpha \forall x \in E \implies x \geq -\alpha \forall x \in E \implies -\alpha$ is a lower bound of E .

2): If $\beta > -\alpha$ then β is not a lower bound of E .

Now $\beta > -\alpha \implies -\beta < \alpha$. So $-\beta$ is not an upper bound of F (since $\alpha = \sup F$). So $\exists -x \in F$ such that $-\beta < -x$, where $x \in E$. Thus $\beta > x$ for some x in E and therefore β is not a lower bound of E .

Exercises 1.5 (page 30):

Problems:

3. If $r \neq 0$ is a rational number and x is an irrational number, then prove that $r+x$ and rx are irrational. Now $r \neq 0 \in \mathbb{Q} \implies r = \frac{a}{b}$, where $a \neq 0, b \neq 0 \in \mathbb{Z}$.

1) Assume $r+x$ is rational. $\implies r+x = \frac{c}{d}$, where $c \neq 0, d \neq 0 \in \mathbb{Z} \implies x = \frac{c}{d} - r$. Then $x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$. Since $bc-ad, bd \neq 0 \in \mathbb{Z}$, x is rational. But it is given that x is irrational, a contradiction. Hence $r+x$ is irrational.

2) Assume rx is rational. $\implies rx = \frac{c}{d}$, where $c \neq 0, d \neq 0 \in \mathbb{Z} \implies x = \frac{c}{d} \cdot \frac{1}{r}$. Then $x = \frac{c}{d} = \frac{bc}{ad}$. Since $bc \neq 0, ad \neq 0 \in \mathbb{Z}$, x is rational, a contradiction to the given that x is irrational. Hence rx is irrational.

6 a. Prove that between any two rational numbers there exists an irrational number.

Let x and y be rational numbers such that $x < y$. Then $\sqrt{2}x < \sqrt{2}y$. Now by the fact that there exist infinitely many rational numbers between any two real numbers, $\exists r \neq 0 \in \mathbb{Q}$ such that $\sqrt{2}x < r < \sqrt{2}y$. Then $x < \frac{r}{\sqrt{2}} < y$. Now since $r \neq 0 \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, $\frac{r}{\sqrt{2}}$ is irrational. Thus $\frac{r}{\sqrt{2}}$ is the required irrational number between the two rational numbers x and y .

6 b. Prove that between any two real numbers there exists an irrational number.

Let $x, y \in \mathbb{R}$ such that $x < y$. Prove: \exists an irrational number z such that $x < z < y$. From the fact that "there exist infinitely many rationals between any two real numbers", \exists rationals r and r' , with $r < r'$, between x and y . Now from 6 a. above, \exists irrational z between r and r' . Thus $x < r < z < r' < y$. Hence \exists irrational z such that $x < z < y$.

7. If $x > 0$, then prove that there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.

Let $x > 0$. Then $\frac{1}{x}$ is not an upper bound of \mathbb{N} (since \mathbb{N} is not bounded above). Thus \exists an $n \in \mathbb{N}$ such that $\frac{1}{x} < n \implies \exists$ an $n \in \mathbb{N}$ such that $\frac{1}{n} < x$. Now $m < 2^m \forall m \in \mathbb{N} \implies \frac{1}{2^n} < \frac{1}{n}$. Thus $\exists n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.