Math 553 Handout #1 Fall 2009

1. Let E be a nonempty subset of $\mathbb{R}$.
   - (a) The number $M \in \mathbb{R}$ is an upper bound of E, if $x \leq M \; \forall x \in E$.
   - (b) The number $M \in \mathbb{R}$ is not an upper bound of E, if $\exists x \in E$ such that $x > M$.
   - (c) The number $m \in \mathbb{R}$ is a lower bound of E, if $x \geq m \; \forall x \in E$.
   - (d) The number $m \in \mathbb{R}$ is not a lower bound of E, if $\exists x \in E$ such that $x < m$.
   - (e) E is said to be bounded above, if it has an upper bound.
   - (f) E is said to be bounded below, if it has a lower bound.
   - (g) $\alpha =$ sup E $\iff$
     (1) α is an upper bound of E.
     (2) If β is an upper bound of E, then $\beta \geq \alpha$.
   - (h) $\alpha =$ inf E $\iff$
     (1) α is a lower bound of E.
     (2) If β is a lower bound of E, then $\beta \leq \alpha$.
2. The Completeness Axiom (The Supremum Property of $\mathbb{R}$):
   Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.
3. For a real number x, the absolute value of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x > o \\ -x, & \text{if } x \leq 0 \end{cases}$$

4. The absolute value function has the following properties:
   - (a) $|x| \geq 0 \; \forall x \in \mathbb{R}$.
   - (b) $|x| = 0$ if and only if $x = 0$.
   - (c) For every real number x, $|-x| = |x|$.
   - (d) For all real numbers x and y, $|xy| = |x||y|$.
   - (e) Let $a > 0$. Then $|x| \leq a \iff -a \leq x \leq a$ and
     $|x| < a \iff -a < x < a$.
   - (f) For every real number x, $-|x| \leq x \leq |x|$.
   - (g) Triangle Inequality: For all real numbers x and y, $|x + y| \leq |x| + |y|$
   - (h) Reverse Triangle Inequality: For all real numbers x and y, $||x| - |y|| \leq |x - y|$.

Use $\epsilon - \delta$ definition to prove:

1. $\underset{x\to -2}{lim} 3x+5=-1$.
   Let $\epsilon > 0$. Find $\delta > 0$ such that if $|x-(-2)| < \delta$, then $|3x+5-(-1)| < \epsilon$.
   Now $|3x+5-(-1)| = |3x+6| = 3|x+2| < \epsilon \implies |x+2| < \frac{\epsilon}{3}$.
   Choose $\delta = \frac{\epsilon}{3}$.
   Verification: If $0 < |x+2| < \delta$ then $|3x+5-(-1)| = |3x+6| = 3|x+2| < 3\delta = 3\frac{\epsilon}{3} = \epsilon$.

2. $\underset{x\to -1}{lim} 2x^2-3x-4=1$.
   Let $\epsilon > 0$. Find $\delta > 0$ such that if $|x-(-1)| < \delta$, then $|2x^2-3x-4-1| < \epsilon$.
   Now $|2x^2-3x-4-1| = |2x^2-3x-5| = |x+1||2x-5| \leq |x+1|(2|x|+5)$ (Bound $|x|$: $|x+1| < 1 \implies -1 < x+1 < 1 \implies -2 < x < 0 \implies |x| < 2$)$< |x+1|(2(2)+5) = 9|x+1| < \epsilon$. $\implies |x+1| < \frac{\epsilon}{9}$.
   Choose $\delta =$ Min $\{1, \frac{\epsilon}{9}\}$.
   Verification: If $0 < |x-(-1)| < \delta$; that is, $0 < |x+1| < \delta$ then $|2x^2-3x-4-1| = |2x^2-3x-5| = |x+1||2x-5| \leq |x+1|(2|x|+5)$ (since $|x+1| < \delta \leq 1 \implies |x| < 2$)$< |x+1|(2(2)+5) = 9|x+1| < 9\frac{\epsilon}{9} = \epsilon$.

3. $\underset{x\to -1}{lim} x^2-x+1=3$.
   Let $\epsilon > 0$. Find $\delta > 0$ such that if $|x-(-1)| = |x+1| < \delta$, then $|(x^2-x+1)-3| < \epsilon$.
   Now $|x^2-x+1-3| = |x^2-x-2| = |x+1||x-2| \leq |x+1|(|x|+2)$(Bound $|x|$: $|x+1| < 1 \implies -1 < x+1 < 1 \implies -2 < x < 0 \implies |x| < 2$) $< |x+1|(2+2) = 4|x+1| < \epsilon$. $\implies |x+1| < \frac{\epsilon}{4}$.
   Choose $\delta =$ Min $\{1, \frac{\epsilon}{4}\}$.
   Verification: If $0 < |x-(-1)| = |x+1| < \delta$; that is, $0 < |x+1| < \delta$ then $|(x^2-x+1)-3| = |x^2-x-2| = |x+1||x-2| \leq |x+1|(|x|+2)$ (since $|x+1| < \delta \leq 1 \implies |x| < 2$)$< |x+1|(2+2) = 4|x+1| < 4\frac{\epsilon}{4} = \epsilon$.

4. $\underset{x\to 2}{lim} \frac{(x^2+2x+2)}{(x+2)} = \frac{5}{2}$.
   Let $\epsilon > 0$. Find $\delta > 0$ such that if $|x-2| < \delta$, then $|\frac{(x^2+2x+2)}{(x+2)} - \frac{5}{2}| < \epsilon$.
   Now $|\frac{(x^2+2x+2)}{(x+2)} - \frac{5}{2}| = |\frac{2x^2+4x+4-5x-10}{2(x+2)}| < |\frac{2x^2-x-6}{x+2}| = |\frac{(x-2)(2x+3)}{x+2}| \leq |x-2|(2|x|+3)\frac{1}{|x+2|}$(Bound $|x+2|$ (away from 0; that is, δ must be smaller than $|2-(-2)| = 4$): $|x-2| < 1 \iff -1 < x-2 < 1 \iff 1 < x < 3$ (Note $|x| < 3$) $\iff 3 < x+2 < 5 \iff \frac{1}{5} < \frac{1}{x+2} < \frac{1}{3}$. Thus $|\frac{1}{x+2}| < \frac{1}{3}$) $< |x-2|(2(3)+3)\frac{1}{3} = 3|x-2| < \epsilon$. So $|x-2| < \frac{\epsilon}{3}$.
   Choose $\delta =$ Min $\{1, \frac{\epsilon}{3}\}$.
   Verification: if $0 < |x-2| < \delta$ then $|\frac{(x^2+2x+2)}{(x+2)} - \frac{5}{2}| \leq |x-2|(2|x|+3)\frac{1}{|x+2|}$ (since $|x-2| < \delta \leq 1$, $|x| < 3$, $\frac{1}{|x+2|} < \frac{1}{3}$) $< |x-2|(2(3)+3)\frac{1}{3} = 3|x-2| < 3\delta = 3\frac{\epsilon}{3} = \epsilon$.

Theorems: 4.1.3
Let E be a subset of $\mathbb{R}$, p a limit point of E, and f a function defined on E. Then $\underset{x\to p}{lim} f(x) = L \iff \underset{n\to\infty}{lim} f(p_n) = L$, for every sequence $\{p_n\}$ in E, with $p_n \neq p$ for all n, and $\underset{n\to\infty}{lim} p_n = p$.
Use Theorem 4.1.3 to prove the statements of Problem 11 (Section 4.1)/ HW#6.

Corollary:
Let E be a subset of $\mathbb{R}$, p a limit point of E, and f a function defined on E. Suppose that there are sequences $\{p_n\}$ in E, with $p_n \neq p$ for all n, and $\{q_n\}$ in E, with $q_n \neq p$ for all n, that converge to p such that $\{f(p_n)\}$ converges to L_1 and $\{f(q_n)\}$ converges to L_2. If $L_1 \neq L_2$, then the function f does not have a limit at p.
Use the above corollary to prove the statements of Problems 1 and 2
(the last two problems given in Homework #6).