Math 331 Test # 1 Material Winter 2012

Do the following.
SECTION 2.1 EXERCISES (page 74) Problems 1-5, 7, 9-14, 18, 20.
Answers. 3. nullity(T)=0, rank(T)=2. Thus, T is 1-1 but not onto.
11. T(8,11) = (5,-3,16)
18. T(a,b)= (b,0), N(T)=R(t) = span {(0,1)}.
Answers to the rest of the problems are given in the text.
Some of these problems are given below.

1. Let V and W be vector spaces (over the field F).
   (a) Please see your notes for the reasons for the steps in the following proofs.
   Prove that if $T : V \longrightarrow W$ is a linear transformation then $T(0_V) = 0_W$.
   $T(00_V) = 0T(0_V) = 0_W$.
   (b) Prove that $T : V \longrightarrow W$ is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and for all $c \in F$.
   ($\Longrightarrow$): Assume T is linear and let $c \in F$ and $x, y \in V$. Then $T(cx + y) = T(cx) + T(y) = cT(x) + T(y)$. ($\Longleftarrow$): Assume $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and for all $c \in F$. Let $c \in F$ and $x, y \in V$.
   (1) $T(x + y) = T(1x + y) = 1T(x) + T(y) = T(x) + T(y)$.
   (2) $T(cx) = T(cx + 0_V) = cT(x) + T(0_V) = cT(x) + 0_W = cT(x)$.

2. Prove that T is a linear transformation, and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.
   (a) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, a_3)$.
   See your notes
   (b) The linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is given by $T(x_1, x_2, x_3, x_4) = (x_1+x_4, x_2+x_3+x_4, 0)$.
   First, verify that T is linear. $N(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | T((x_1, x_2, x_3, x_4) = (0, 0, 0)\}$
   $= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | (x_1+x_4, x_2+x_3+x_4, 0) = (0, 0, 0)\} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1+x_4 = 0, x_2 + x_3 + x_4 = 0\}$. Now solve the system $x_1 + x_4 = 0, x_2 + x_3 + x_4 = 0$.
   The reduced row-echelon form of $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the matrix A itself.
   Since there are two free variables, $\dim(N(T)) = \text{nullity}(A) = 2$. So $x_1+x_4 = 0$, $x_2+x_3+x_4 = 0$ $\Longrightarrow$ $x_1 = -x_4$ and $x_2 = -x_3 - x_4$.
   Let $x_3 = 1$ and $x_4 = 0$. Then $x_1 = 0$ and $x_2 = -1$. Thus $(0, -1, 1, 0)$ is a basis vector. The other basis vector is obtained by letting, $x_3 = 0, x_4 = 1$. Then $x_1 = -1, x_2 = -1$ and the second basis vector is: $(-1, -1, 0, 1)$. A basis for $N(T)$ (the null space of A) is $\{(0, -1, 1, 0), (-1, -1, 0, 1)\}$.

   By Theorem 2.2, since a basis for $\mathbb{R}^4$ is $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. $\mathbb{R}^4$= span$\{T((1, 0, 0, 0), T((0, 1, 0, 0), T(0, 0, 1, 0), T(0, 0, 0, 1)\} = \{(1, 0, 0), (0, 1, 0), (0, 1, 0), (1, 1, 0)\}$.
   Thus, a basis for $R(T)$ is $\{T((1, 0, 0, 0), T((0, 1, 0, 0), T(0, 0, 0, 1)\}$. Hence rank(T) = 2.
   Neither one-one nor onto. Now verify Theorem 2.3.
   (c) $T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.
   Verification that T is linear: Let $c \in \mathbb{R}$ and $f(x), g(x) \in P_2(\mathbb{R})$. Then $T(cf(x) + g(x)) = x(cf(x) + g(x)) + (cf(x) + g(x))' = x(cf(x) + g(x)) + cf'(x) + g'(x) = c(xf(x) + f'(x)) + (xg(x) + g'(x)) = cT(f(x)) + T(g(x))$.
   Since $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$, $R(T) = \text{span}\{T(1), T(x), T(x^2)\} = \text{span}\{x, x^2+1, x^3+2x\}$.
   However, $\{x, x^2+1, x^3+2x\}$ is linearly independent. So, a basis for $R(T)$ is $\{x, x^2+1, x^3+2x\}$ and rank(T) = dim(R(T)=3. By Theorem B, $\dim(P_2(R))$ = nullity(T)+rank(T). Thus, 3= nullity(T)+3. Hence, nullity(T)= 0. So, N(T)={0}.
   T is 1-1 but not onto.

3. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a function. State why T is not linear.
   (a) $T(a_1, a_2) = (\sin a_1, 0)$. Let $a_1 = \frac{\pi}{2}$ and $a_2 = 0$; $b1 = \frac{\pi}{2}$, $b_2 = 0$. Now verify that $T(a_1, a_2) + T(b_1, b_2) \neq T(a_1 + b_1, a_2 + b_2)$.
   (b) $T(a_1, a_2) = (a_1 + 1, a_2)$.
   $T(0, 0) \neq (0, 0)$

4. Prove that there exists a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?
   By Theorem 2.6, since $\{(1, 1), (2, 3)\}$ is a basis for $\mathbb{R}^2$, T is linear. (8,11) = 2(1,1)+3(2,3).

5. Let V and W be vector spaces, let $T : V \longrightarrow W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.
   Assume $\exists c_1, c_c2, \ldots, c_k \in F$ such that
   $c_1v_1+c_2v_2+\ldots+c_kv_k = 0_V$. Then $T(c_1v_1+c_2v_2+\ldots+c_kv_k) = T(0_V)$. So, $c_1w_1+c_2w2+\ldots+c_kw_k = 0_W$. But $\{w_1, w_2, \ldots, w_k\}$ is linearly independent. So, $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

6. Let V and W be vector spaces and $T : V \longrightarrow W$ be linear.
   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
   ($\Longrightarrow$): Assume T is 1-1 and $\{v_1, v_2, \ldots, v_k\}$ be linearly independent. Now $c_1T(v_1)+c_2T(v_2)+\ldots+c_kT(v_k) = 0_W$ gives $T(c_1v_1+c_2v_2+\ldots+c_kv_k) = T(0_V) \implies c_1v_1+c_2v_2+\ldots+c_kv_k = 0_V$, since T is 1-1. But $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. So, $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. ($\Longleftarrow$): Assume T carries linearly independent subsets of V onto linearly independent subsets of W. Proof By contradiction. Assume T is not 1-1. Then, by Theorem D above, $N(T) \neq \{0_V)\}$. Then $N(T)$ has a nonempty basis, say $\{v_1, v_2, \ldots, v_k\}$. Now $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent subset of V (since N(T) $\subseteq$ V) but $\{T(v_1), T(v_2), \ldots, T(v_n)\} = \{0_W\}$ (since $v_i \in N(T)$, $i = 1, 2, \ldots, v_n$) is linearly dependent (any subset of a vector space that contains the zero vector is linearly dependent).
   (b) Suppose that T is one-to-one and S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.
   Let S=$\{\{v_1, v_2, \ldots, v_k\}$. Then $T(S) = \{T(v_1), T(v_2), \ldots + T(v_k)\}$.
   $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0_V \iff c_1T(v_1) + c_2T(v_2) + \ldots + c_kT(v_k) = 0_W$.
   Now, complete the p.roof
   (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.
   $T(\beta) = W$ (since T is onto) $T(\beta)$ is LI by (b)

7. Let V and W be finite-dimensional vector spaces and $T : V \longrightarrow W$ be linear.
   (a) Prove that if dim(V) < dim (W), then T cannot be onto.
   rank(T) < dim(V) < dim (W)
   (b) Prove that if dim(V) > dim (W), then T cannot be one-to-one.
   dim(V) > dim (W) $\geq$ rank(T) (Definition of R(T)) $\implies$ dim(V) > rank(T). By Theorem 2.3, nullity(T) $\geq$ 1. So N(T) $\neq \{0_V\}$. By Theorem 2.4, T is not 1-1

8. Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $N(T) = R(T)$.
   T(a,b)= (b,0), N(T)=R(t) = span (0,1).
   Do the following.
   SECTION 2.2 EXERCISES (page 84) Problems 1-5, 6-8
   Answers. 2(b) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix}$. 2(e) $\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix}$. 4. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \end{bmatrix}$. 5 (c) $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$.

5(d) $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$. 5(f) $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$. 5(g) $[a]$.
For 6,7, and 8, see THEOREMS at the end of this document.
Answers to the rest of the computational problems are given in the text (page 573).

9. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by $T(a_1, a_2, a_3) = (a_1 + a_2 - a_3, 2a_1 + a_3)$.
   (a) Let β and γ be the standard bases for R^3 and R^2, respectively.
   Find $[T]_\beta^\gamma$.

   (Answer: Note $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $\gamma = \{(1,0),(0,1)\}$. $[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.)

   (b) Let $\beta = \{(1,0,-1),(1,1,1),(1,0,0)\}$ and $\gamma = \{(1,1),(1,0)\}$ be bases for R^3 and R^2, respectively. Find $[T]_\beta^\gamma$.
   (Answer: $[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -2 & -1 \end{bmatrix}$.)

10. Let $T : P_2(\mathbb{R}) \longrightarrow M_{2\times 2}$ be given by $T(a + bx + cx^2) = \begin{bmatrix} a & -c \\ -c & a-b \end{bmatrix}$. Let β and γ be the standard bases of $P_2(\mathbb{R})$ and $M_{2\times 2}$, respectively.

11. Find $[T]_\beta^\gamma$.
    (Answer: Note $\beta = \{1, x, x^2\}$ and $\gamma = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.
    $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.)

12. If $f(x) = 2 - x + 5x^2$, find $[f(x)]_\beta$.
    (Answer: $[f(x)]_\beta = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.)

Do the following.
SECTION 2.4 EXERCISES (page 106) Problems 2, 3, 14
Answers.
2(a). R(T) = span$\{T(1,0), T(0,1)\}$ =span$\{(1,0,3),(-2,1,4)\} \neq \mathbb{R}^3$. Thus, T is not onto. So T is not invertible.
2(b). R(T) = span$\{T(1,0), T(0,1)\}$ =span$\{(3,0,4),(-1,1,0)\} \neq \mathbb{R}^3$. Thus, T is not onto. So T is not invertible.
2(c). $T(a_1, a_2, a_3) = (0,0,0) \iff (3a_1 - 2a_3, a_2, 3a_1 + 4a_2) = (0,0,0) \iff a_1 = 0, a_2 = 0, a_3 = 0$. Hence $N(T) = (0,0,0)$. Thus, T is 1-1. By Theorem 2.5 (page 71), T is onto. Therefore, T is invertible.
Formula for T^{-1}:
Let $(a,b,c) \in \mathbb{R}^3$. Solve $T(a_1, a_2, a_3) = (a,b,c)$ for a_1, a_2, and a_3. Now $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. So $(3a_1 - 2a_3, a_2, 3a_1 + 4a_2) = (a,b,c)$. Thus, $3a_1 - 2a_3 = a, a_2 = b, 3a_1 + 4a_2 = c$.
The row-reduced echelon form of $\begin{bmatrix} 3 & 0 & -2 & a \\ 0 & 1 & 0 & b \\ 3 & 4 & 0 & c \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 0 & -2 & \frac{c-4b}{3} \\ 0 & 1 & 0 & b \\ 3 & 4 & 0 & \frac{c-a-4b}{2} \end{bmatrix}$ So, $a_1 = \frac{c-4b}{3}$, $a_2 = b$, $a_3 = \frac{c-a-4b}{2}$. Hence, the formula for T^{-1} is $T^{-1}(a,b,c) = (\frac{c-4b}{3}, b, \frac{c-a-4b}{2})$.
Problem 2(d). N(T): $T(a + bx + cx2) = (a + bx + cx2)' = b + 2cx$ So $T(a + bx + cx2) = 0 \iff b + 2cx = 0 \iff b = 0$ and $c = 0$. Thus, $N(T) = \{a | a \in \mathbb{R}\}$ = span$\{1\}$. By Theorem 2.4 (page 71), T is not 1-1. Hence, T is not invertible.

Problem 2(e). R(T) = span$\{T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})\}$ = span $\{1, 2x, x^2, x^2\}$= span$\{1, 2x, x^2\} = P_2(\mathbb{R})$. Thus T is onto. By Theorem 2.2 (page 70), nullity(T)=1. Hence, by Theorem 2.4 (page 71), T is not 1-1.
2(f). $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff a+b=0, a=0, c=0, c+d=0 \iff a=0, b=0, c=0, d=0$. Thus, $N(T) = \{0\}$. By Theorem 2.4 (page 71), T is 1-1. By Theorem 2.5 (page 71), T is onto. Hence T is invertible.
Formula for T^{-1}: $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Longrightarrow \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Longrightarrow c+b = e, a = f, c = g, c+d = h$. Now solve $c+b = e, a = f, c = g, c+d = h$ for a,b,c,d. We have, a=f,c=g, b=e-g, d=h-g. Hence the formula for T^{-1} is given by:
$T(\begin{bmatrix} f & e-g \\ g & h-g \end{bmatrix}) = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.
Problem 3(a). $\dim(F^3) = 3 \neq 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), F^3 and $P_3(F)$ are not isomorphic.
Problem 3(b). $\dim(F^4) = 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), F^4 and $P_3(F)$ are isomorphic.
Problem 3(c). $\dim(M_{2\times2}) = 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), $M_{2\times2}$ and $P_3(F)$ are isomorphic.
Problem 3(d). $V = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R}) | tr(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R}) | a+d = 0\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R}) | d = -a\} = \{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} | a, b, c \in \mathbb{R}\}$.
Thus $\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \}0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$ is a basis for V. Now dim(V) = $3 \neq 4 = \dim(\mathbb{R}^4)$. Hence, By Theorem 2.19 (page 103), V and ℝ4 are not isomorphic.

13. For each of the following linear transformations T, determine whether T is invertible and justify your answer. It T is invertible, then find a formula for T.

    (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
    Problem 2(a)/Exercises page 106. Solved above.

    (b) $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(f(x)) = xf'(x)$.
    12(b) N(T): $T(a+bx+cx2+dx3) = (a+bx+cx2+dx3)' = b+2cx+3dx2$. $T(a+bx+cx2+dx3) = 0 \iff b+2cx+3dx2 = 0 \iff b = 0, c = 0, d = 0$. Thus $N(T) = \{a+bx+cx2+dx3 \in P_3(\mathbb{R}) | b = 0, c = 0, d = 0\} = \{a | a \in \mathbb{R}\} = \mathbb{R}$. By Theorem 2.4 (page 71), since $N(T) \neq \{0\}$, T is not 1-1. Hence, T is not invertible.

    (c) $T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$ defined by $T(f(x)) = (f(0), f(1), f(2))$.
    12(c) $f(x) = a + bx + cx^2$. $f(0) = a, f(1) = a+b+c, f(2) = a+2b+4c$.
    $T(f(x)) = (0,0,0) \iff (f(0), f(1), f(2)) = (0,0,0) \iff (a, a+b+c, a+2b+4c) = (0,0,0) \iff a = 0, a+b+c = 0, a+2b+4c = 0 \iff a = 0, b = 0, c = 0$. Thus, $N(T) = \{0\}$. $\iff a+b = 0, a = 0, c = 0, c+d = 0 \iff a = 0, b = 0, c = 0, d = 0$. Thus, $N(T) = \{0\}$. By Theorem 2.4 (page 71), T is 1-1. By Theorem 2.5 (page 71), T is onto. Hence T is invertible.
    Formula for T^{-1}:
    $T^{-1}(a,b,c) = a + (\frac{-3}{2}a + 2b - \frac{1}{2}c)x + (\frac{1}{2}a - b + \frac{1}{2}c)x2.$

    (d) $T : V \longrightarrow V$, dim(V)=4, and with respect to the basis $\{v_1, v_2, v_3, v_4\}$, T is given by $T(v_1) = v_2$, $T(v_2) = v_1$, $T(v_3) = v_4$, $T(v_4) = v_3$.
    R(T) = span$\{T(v_1), T(v_2), T(v_3), T(v_4)\}$ = span$\{v_2, v_1, v_4, v_3\}$= V. Thus, T is onto, by the definition of an onto mapping. By Theorem 2.5 (page 71), T is also 1-1. Hence T is invertible.
    Formula for T^{-1}: $T^{-1}(v_1) = v_2$,$T^{-1}(v_2) = v_1$, $T^{-1}(v_3) = v_4$, $T^{-1}(v_4) = v_3$.

Note $T = T^{-1}$.

14. Which of the following pairs of vector spaces are isomorphic? Justify your answer.

    (a) $V = \mathbb{R}^4$, $W = \{f(x) \in P_4(\mathbb{R}) | f(0) = 0\}$.
    Note $W = \{ax + bx2 + cx3 + dx4 | a, b, c, d \in \mathbb{R}\} = \text{span}\{x, x2, x3, x4\}$.
    Now dim(V) = dim($\mathbb{R}^4$) = 4 = dim(W). Hence, By Theorem 2.19 (page 103), V and W are isomorphic.

    (b) $V = P_5(\mathbb{R})$, W= $M_{2\times3}(\mathbb{R})$.
    13(b) dim(V) = dim($P_5(\mathbb{R})$ = 6 =dim($M_{2\times3}(\mathbb{R})$ = dim(W). By Theorem 2.19, (page 103), V and W are isomorphic.

    (c) $V = \{A \in M_{2\times2}(\mathbb{R}) | tr(A) = 0\}$, $W = \mathbb{R}^4$.

    See Problem 3(d) / Exercises (page 106) (solved above).

THEOREMS

1. Theorem 2.2: Let V and W be vector spaces and $T : V \longrightarrow W$ be linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then
$R(T) = \text{span}(T(\beta)) = \text{span } \{T(v_1), T(v_2), \ldots, T(v_n)\}$.
Let $w \in R(T)$. Then $\exists v \in V$ such that $T(v) = w$. Since $v \in V$ and β is a basis for V, $\exists c_1, c_2, \ldots, c_n \in F$ such that $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$. From $T(v) = w$, we have $w = T(c_1v_1 + c_2v_2 + \ldots + c_nv_n)$. Since T is linear, $w = c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n)$. Thus $w \in\text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$ and $R(T) \subseteq \text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$. Let $w \in$ span $\{T(v_1), T(v_2), \ldots, T(v_n)\}$. Then $\exists c_1, c_2, \ldots, c_n \in F$ such that $w = c_1T(v_1)+c_2T(v_2)+\ldots+c_nT(v_n)$. Since T is linear, $w = T(c_1v_1+c_2v_2+\ldots+c_nv_n)$. Thus $w \in R(T)$ and $\text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\} \subseteq R(T)$. Hence $R(T) = \text{span } \{T(v_1), T(v_2), \ldots, T(v_n)\}$.

2. Theorem 2.3 (Dimension Theorem): Let V and W be vector spaces and $T : V \longrightarrow W$ be linear. If V is finite-dimensional, then
nullity(T) + rank(T) = dim(V).
Case (1). Assume $N(T) = \{0_V\}$; that is, nullity(T)=0. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. By Theorem 2.2 above, $R(T) = \text{span } \{T(v_1), T(v_2), \ldots, T(v_n)\}$. Suppose $\exists c_1, c_2, \ldots, c_n \in F$ such that $c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n) = 0_W$. Since T is linear, $T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = 0_W$. Thus $c_1v_1 + c_2v_2 + \ldots + c_nv_n \in N(T) = \{0_V\}$. So $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0_V$. But $\{v_1, v_2, \ldots, v_n\}$, being a basis for V, is linearly independent. Hence $c_1 = 0, c_2 = 0, \ldots, c_n = 0$ and $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent. So, $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for $R(T)$. Therefore, rank(T) = dim(R(T))=n. Thus, n= dim(V)=0+n = nullity(T) + rank(T).
Case (2). Assume $N(T) = V$; that is, nullity(T)=n. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. By Theorem 2.2 above, $R(T) = \text{span } \{T(v_1), T(v_2), \ldots, T(v_n)\}$. Thus, $R(T) = \text{span } \{0_V\} = \{0_V\}$. So rank(T)=dim(R(T))=0. Hence n= dim(V)=n+0 = nullity(T) + rank(T).
Case (3). Assume nullity(T)=k, where $1 \leq k < n$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $N(T)$. Now $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent subset of V and every linearly independent subset of V can be extended to give a basis for V. So $\exists v_{k+1}, v_{k+2}, \ldots, v_n \in V$ such that $\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n\}$ is a basis for V. By Theorem 2.2, R(T) = span $\{T(v_1), T(v_2), \ldots, T(v_k), T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$. But $\{T(v_1), T(v_2), \ldots, T(v_k), T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} = \{0_V, 0_V, \ldots, 0_V, T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} = \{0_V, T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$. Thus, R(T)= $\text{span}\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$. Suppose $existsc_{k+1}, c_{k+2}, \ldots, c_n \in F$ such that $c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \ldots + c_nT(v_n) = 0_W$. Since T is linear, $T(c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \ldots + c_nv_n) = 0_W$. Thus, $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \ldots + c_nv_n \in N(T) = \{0_V\}$. But $\{v_1, v_2, \ldots, v_k\}$ is a basis for $N(T)$. So $\exists b_1, b_2, \ldots, b_k \in F$ such that $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \ldots + c_nv_n = b_1v_1 + b_2v_2 + \ldots + b_kv_k$. Then $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \ldots + c_nv_n - b_1v_1 - b_2v_2 - \ldots - b_kv_k = 0_V$. Since $\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n\}$ is linearly independent, $c_{k+1} = 0, c_{k+2} = 0, \ldots, c_n = 0$. So $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \ldots + c_nv_n = 0_V$. Therefore,

$\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is linearly independent and $\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is a basis for R(T). So, rank(T) = dim(R(T))=n-k. Hence, dim(V)=n= k + n-k = nullity(T) + rank(T).

3. Theorem 2.4: Let V and W be vector spaces and $T : V \longrightarrow W$ be linear. T is 1-1 if and only if $N(T) = \{0_V\}$.
   ($\Longrightarrow$): Assume T is 1-1. Now $\{0_V\} \subseteq N(T)$, since N(T) is a subspace of V. Let $v \in N(T)$. Then $T(v) = 0_W$. But $T(0_V) = 0_W$. So $T(v) = T(0_V)$. Thus $v = 0_V$, since T is 1-1. So $v \in \{0_V\}$ and $N(T) \subseteq \{0_V\}$. Hence $N(T) = \{0_V\}$.
   ($\Longleftarrow$): Assume $N(T) = \{0_V\}$. Assume $T(v) = T(v')$ for some $v, v' \in V$. Then $T(v) - T(v') = 0_W \Longrightarrow T(v - v') = 0_W$, since T is linear. Thus, $v - v' \in N(T) = \{0_V\}$. So $v - v' = 0_V$. Thus $v = v'$ and T is 1-1.

4. Theorem 2.5: Let V and W be vector spaces of equal (finite) dimension and $T : V \longrightarrow W$ be linear. Then the following conditions are equivalent.
   (a) T is 1-1.
   (b) T is onto.
   (c) rank(T) = dim(V).
   We will prove that (1): (a) $\Longrightarrow$ (b), (2): (b) $\Longrightarrow$ (c), (3): (c) $\Longrightarrow$ (a).
   Proof of (1). Assume T is 1-1. By Theorem 2.4 (page 71) $N(T) = \{0_V\}$. So, nullity(T)=0. By Theorem 2.2, nullity(T)+rank(T)=dim(V). Then rank(T)=dim(V). But dim(v)=dim(W). So rank(T)=dim(W). Now R(T) is a subspace of W and dim(R(T))= rank(T)=dim(W). Hence R(T)=W and T is onto.
   Proof of (2). Assume T is onto. Then R(T)=W (definition). Thus rank(T) = dim(W). But dim(v)=dim(W). So rank(T)=dim(V). Proof of (3). Assume rank(T)=dim(V). By Theorem 2.2, nullity(T)+rank(T)=dim(V). Then nullity(T)=0. So $N(T) = \{0_v\}$. By Theorem 2.4 (page 71), T is 1-1.

5. Let V and W be vector spaces of equal (finite) dimension and $T : V \longrightarrow W$ be linear. Prove that T is invertible if and only if rank(T) = dim(V).
   ($\Longrightarrow$): Assume T is invertible. So T is 1-1 and onto. rank(T) = dim(V). ($\Longleftarrow$): Assume rank(T) = dim(V). By Theorem 2.5 given above, T is both 1-1 and onto. Hence T is invertible.

6. Theorem 2.6: Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n \in W$, there exists exactly one linear transformation $T : V \longrightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.
   Let $v \in V$. Since $v \in V$ and $\{v_1, v_2, \ldots, v_n\}$ is a basis for V i, $\exists c_1, c_2, \ldots, c_n \in F$ such that $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$. Define $T : V \longrightarrow W$ by $T(v) = c_1w_1 + c_2w_2 + \ldots + c_nw_n$.
   (1) Show T is linear. Let $c \in F$ and $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ and $v' = d_1v_1 + d_2v_2 + \ldots + d_nv_n$ be in V. Then $T(cv + v') = T(c(c_1v_1 + c_2v_2 + \ldots + c_nv_n) + d_1v_1 + d_2v_2 + \ldots + d_nv_n) = T((cc_1 + d_1)v_1 + (cc_2 + d_2)v_2 + \ldots + (cc_n + d_n)v_n) = (cc_1 + d_1)w_1 + (cc_2 + d_2)w_2 + \ldots + (cc_n + d_n)w_n$ (definition of T) $= c(c_1w_1 + c_2w_2 + \ldots + c_nw_n) + d_1w_1 + d_2w_2 + \ldots + d_nw_n = cT(v) + T(v')$.
   (2) Note that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.
   (3) Let U be another linear transformation $U : V \longrightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$. Now $U(v) = U(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = c_1U(v_1) + c_2U(v_2) + \ldots + c_nU(v_n)$ (since U is linear) $= c_1w_1 + c_2w_2 + \ldots + c_nw_n$ (definition of U) $= T(c_1v_1 + c_2v_2 + \ldots + c_nv_n)$ (definition of T (see above)) $= T(v)$.
   So $U = T$.

7. (Theorem 2.7 (page 82)) Let V and W be vector spaces over the field F, and let $T, U : V \longrightarrow W$ be linear. Then for all $a \in F$, $aT + U$ is linear.
   Let $v, v' \in V$ and let $c \in F$. Show $(aT + U)(cv + v') = c(aT + U)(v) + (aT + U)(v')$.
   $(aT + U)(cv + v') = (aT)(cv + v') + U(cv + v')$ (sum of mappings) $a(T(cv + v')) + U(cv + v')$ (definition of aT) $= a(cT(v) + T(v')) + cU(v) + U(v')$ (T,U are linear) $= c(aT(v) + U(v)) + aT(v') + U(v')$ (W is a vector space) $= c(aT + U)(v) + (aT + U)(v')$ (definition of aT+U).

8. (Theorem 2.8 (page 82)) Let V and W be finite-dimensional vector spaces over the field F, with ordered bases β and γ, respectively, and let $T, U : V \longrightarrow W$ be linear. Then

   (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.
   Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ be ordered bases for V and W, respectively. For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m$. Thus, $[T(v_j)]_\gamma = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$.

   Now $[T]_\beta^\gamma = [[T(v_1)]_\gamma [T(v_2)]_\gamma \ldots [T(v_j)]_\gamma \ldots [T(v_n)]_\gamma] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$.

   We will represent this as $[T]_\beta^\gamma = (a_{ij})$. For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $U(v_j) = b_{1j}w_1 + b_{2j}w_2 + \ldots + b_{ij}w_i + \ldots + b_{mj}w_m$.

   Thus, $[T(v_j)]_\gamma = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$.

   Now $[T]_\beta^\gamma = [[U(v_1)]_\gamma [U(v_2)]_\gamma \ldots [U(v_j)]_\gamma \ldots [U(v_n)]_\gamma] = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{ij} & \ldots & b_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mj} & \ldots & b_{mn} \end{bmatrix}$.

   We will represent this as $[U]_\beta^\gamma = (b_{ij})$. Now $[T]_\beta^\gamma + [U]_\beta^\gamma = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $(T+U)(v_j) = T(v_j) + U(v_j) = (a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m) + (b_{1j}w_1 + b_{2j}w_2 + \ldots + b_{ij}w_i + \ldots + b_{mj}w_m = (a_{1j}+b_{1j})w_1 + (a_{2j}+b_{2j})w_2 + \ldots + (a_{ij}+b_{ij})w_i + \ldots + (a_{mj}+b_{mj})w_m$ and $[T+U]_\beta^\gamma = (a_{ij} + b_{ij})$.

   (b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma \ \forall a \in F$.
   Let $a \in F$. From the part (a), $a[T]_\beta^\gamma = a(a_{ij}) = (aa_{ij}) = [aT]_\beta^\gamma$.

9. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \longrightarrow F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.
   Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for V. Let $x, y \in V$ and $c \in F$. Show $T(cx+y) = cT(x) + T(y)$.
   Since β is a basis for V, and $x, y \in V$, $\exists c_1, c_2, \ldots, c_n \in F$ such that $x = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ and $\exists d_1, d_2, \ldots, d_n \in F$ such that $y = d_1v_1 + d_2v_2 + \ldots + d_nv_n$. Note $cx = cc_1v_1 + cc_2v_2 + \ldots + cc_nv_n$ and $cx + y = (cc_1 + d_1)v_1 + (cc_2 + d_2)v_2 + \ldots + (cc_n + d_n)v_n$. Thus, $T(cx+y) = [x+y]_\beta =$

$$\begin{bmatrix} cc_1+d_1 \\ cc_2+d_2 \\ \vdots \\ cc_n+d_n \end{bmatrix} = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = c\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = c[x]_\beta + [y]_\beta = cT(x) + T(y)$$

10. (Theorem 2.9 (page 86)) Let V, W, and Z be vector spaces over the field F, and let $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ be linear. Then $UT : V \longrightarrow Z$ is linear.
    Let $c \in F$, and $x, y \in V$.
    Then $(UT)(cx+y) = U(T(cx+y))$(definition of the composite UT) $= U(cT(x)+T(y))$(T is linear) $= cU(T(x)) + U(T(y)))$ ($U : W \longrightarrow Z$ is linear and $T(x), T(y) \in W$) $= c(UT)(x) + (UT)(y)$ (definition of the composite UT)

11. (Theorem 2.17 (page 100)) Let V and W be vector spaces over the field F, and let $T : V \longrightarrow W$ be invertible and linear. Then $T^{-1} : W \longrightarrow V$ is linear.
    Let $w, w' \in W$ and let $c \in F$. Show $T^{-1}(cw + w') = cT^{-1}(w) + T^{-1}(w')$.
    Now $w, w' \in W$ and $T : V \longrightarrow W$ is onto $\Longrightarrow$ $\exists v, v' \in V$ such that $T(v) = w$ and $T(v') = w'$.
    Then $T^{-1}(cw+w') = T^{-1}(cT(v)+T(v') = T^{-1}(T(cv+v')$ (since T is linear) $= (T^{-1}T)(cv+v') = 1_V(cv + v')$ ($T^{-1}T = 1_V$, the identity mapping on V) $= cv + v'$ ($cv + v' \in V$, definition of 1_V) $= cT^{-1}(w) + T^{-1}(w')$ (since $T(v) = w$ and $T(v') = w'$).

12. (Lemma (page 101)) Let V and W be vector spaces and let $T : V \longrightarrow W$ be linear and invertible. Then

    (a) V is finite-dimensional if and only if W is finite-dimensional.
    ($\Longrightarrow$): Assume V is finite-dimensional and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then, by Theorem 2.2 (page 68), R(T) = span(β). But R(T)=W, since T is onto. So W=span(β).But a basis B of W is a subset of the spanning set β. Thus $dim(W) = |B| \leq |\beta| = n$. Hence, W is finite-dimensional.
    ($\Longleftarrow$): Assume W is finite-dimensional and let $\gamma = \{w_1, w_2, \ldots, w_n\}$ be a basis for W. Since T is invertible, $T^{-1} : W \longrightarrow V$ is linear, 1-1, and onto. Then, by Theorem 2.2 (page 68), $R(T^{-1})$= span(γ). But $R(T^{-1}) = V$, since T^{-1} is onto. So V=span(γ).But a basis A of V is a subset of the spaning set γ. Thus $dim(V) = |A| \leq |\gamma| = n$. Hence, V is finite-dimensional.

    (b) If V and W are finite-dimensional, then dim(V) = dim(W).
    Since T is onto, R(T)=W. Thus rank(T) = dim(W). But T is 1-1 $\Longrightarrow$ nullity(T) =0 (Theorem 2.4 (page 71)). By Theorem 2.4, rank(T)=dim(V). Hence rank(T)=dim(V)=dim(W).

13. (Theorem 2.19 (page 103)) Let V and W be finite-dimensional vector spaces over the field F. Then V is isomorphic to W if and only if dim(V) = dim(W).
    ($\Longrightarrow$): V is isomorphic to W $\Longrightarrow$ $\exists$ an invertible linear transformation $T : V \longrightarrow W$. Then by the above, dim(V)=dim(W). ($\Longleftarrow$): Assume dim(V) = dim(W)=n and let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ be bases for V and W, respectively. By Theorem 2.6 (page 72), there exists a linear transformation $T : V \longrightarrow W$ defined by
    $T(v_1) = w_1, T(v_2) = w_2, \ldots, T(v_n) = w_n$. By Theorem 2.2, R(T)=span $\{T(v_1), T(v_2), \ldots, T(v_n)\}$= span$\{w_1, w_2, \ldots, w_n\}$. But span($\gamma$)= span$\{w_1, w_2, \ldots, w_n\} = W$, since $\gamma = \{w_1, w_2, \ldots, w_n\}$ is a basis of W. Thus R(T)=W and T is onto. Since dim(V)=dim(W), by Theorem 2.5 (page 71) T is also 1-1. Hence T is an invertible linear transformation.