Math 531 Sample Quiz # 3 Winter 2012

Do the following.
SECTION 2.2 EXERCISES (page 84) Problems 6, 7, 8, 10
SECTION 2.4 EXERCISES (page 106) Problems 2, 3

SECTION 2.4 EXERCISES (page 106)

1. 2(a). R(T) = span$\{T(1,0), T(0,1)\}$ =span$\{(1,0,3),(-2,1,4)\} \neq \mathbb{R}^3$. Thus, T is not onto. So T is not invertible.

2. 2(b). R(T) = span$\{T(1,0), T(0,1)\}$ =span$\{(3,0,4),(-1,1,0)\} \neq \mathbb{R}^3$. Thus, T is not onto. So T is not invertible.

3. 2(c). $T(a_1,a_2,a_3) = (0,0,0) \iff (3a_1-2a_3, a_2, 3a_1+4a_2) = (0,0,0) \iff a_1 = 0, a_2 = 0, a_3 = 0$. Hence $N(T) = (0,0,0)$. Thus, T is 1-1. By Theorem 2.5 (page 71), T is onto. Therefore, T is invertible.
   Formula for T^{-1}:
   Let $(a,b,c) \in \mathbb{R}^3$. Solve $T(a_1,a_2,a_3) = (a,b,c)$ for a_1, a_2, and a_3. Now $T(a_1,a_2,a_3) = (3a_1 - 2a_3, a_2, 3a_1+4a_2)$. So $(3a_1-2a_3, a_2, 3a_1+4a_2) = (a,b,c)$. Thus, $3a_1-2a_3 = a, a_2 = b, 3a_1+4a_2 = c$.
   The row-reduced echelon form of $\begin{bmatrix} 3 & 0 & -2 & a \\ 0 & 1 & 0 & b \\ 3 & 4 & 0 & c \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 0 & -2 & \frac{c-4b}{3} \\ 0 & 1 & 0 & b \\ 3 & 4 & 0 & \frac{c-a-4b}{2} \end{bmatrix}$ So, $a_1 = \frac{c-4b}{3}$, $a_2 = b$, $a_3 = \frac{c-a-4b}{2}$. Hence, the formula for T^{-1} is $T^{-1}(a,b,c) = (\frac{c-4b}{3}, b, \frac{c-a-4b}{2})$.

4. Problem 2(d). N(T): $T(a+bx+cx^2) = (a+bx+cx^2)' = b+2cx$ So $T(a+bx+cx2) = 0 \iff b+2cx = 0 \iff b = 0$ and $c = 0$. Thus, $N(T) = \{a|a \in \mathbb{R}\} =$ span$\{1\}$. By Theorem 2.4 (page 71), T is not 1-1. Hence, T is not invertible.

5. Problem 2(e). R(T) = span$\{T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})\}$ = span $\{1, 2x, x^2, x^2\}$= span$\{1, 2x, x^2\} = P_2(\mathbb{R})$. Thus T is onto. By Theorem 2.2 (page 70), nullity(T)=1. Hence, by Theorem 2.4 (page 71), T is not 1-1.

6. 2(f). $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff a+b = 0, a = 0, c = 0, c+d = 0 \iff a = 0, b = 0, c = 0, d = 0$. Thus, $N(T) = \{0\}$. By Theorem 2.4 (page 71), T is 1-1. By Theorem 2.5 (page 71), T is onto. Hence T is invertible.
   Formula for T^{-1}: $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \implies \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \implies c+b = e, a = f, c = g, c+d = h$. Now solve $c+b = e, a = f, c = g, c+d = h$ for a,b,c,d. We have, a=f,c=g, b=e-g, d=h-g. Hence the formula for T^{-1} is given by:
   $T(\begin{bmatrix} f & e-g \\ g & h-g \end{bmatrix}) = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

7. Problem 3(a). $\dim(F3) = 3 \neq 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), $F3$ and $P_3(F)$ are not isomorphic.

8. Problem 3(b). $\dim(F4) = 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), $F4$ and $P_3(F)$ are isomorphic.

9. Problem 3(c). $\dim(M_{2\times 2}) = 4 = \dim(P_3(F))$. By Theorem 2.19 (page 103), $M_{2\times 2}$ and $P_3(F)$ are isomorphic.

10. $V = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) | tr(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) | a+d = 0\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) | d = -a\} = \{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} | a, b, c \in \mathbb{R}\}$.

Thus $\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \}0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$ is a basis for V. Now dim(V) $= 3 \neq 4 = \dim(\mathbb{R}^4)$. Hence, By Theorem 2.19 (page 103), V and $\mathbb{R}4$ are not isomorphic.

1. (Theorem 2.7 (page 82)) Let V and W be vector spaces over the field F, and let $T, U : V \longrightarrow W$ be linear. Then for all $a \in F$, $aT + U$ is linear.
   Let $c \in F$, and $x, y \in V$.
   Then $(aT+U)(cx+y) = (aT)(cx+y)+U(cx+y)$(definition of the sum of mappings) $= a(T(cx+y)) + U(cx+y)$(definition of (aT)) $= a(cT(x) + T(y)) + cU(x) + U(y)$ (T and U are linear) $= c(aT + U)(x) + (aT + U)(y)$ (re-arranging in W)

2. (Theorem 2.8 (page 82)) Let V and W be finite-dimensional vector spaces over the field F, with ordered bases β and γ, respectively, and let $T, U : V \longrightarrow W$ be linear. Then

   (a) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.
   Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ be ordered bases for V and W, respectively. For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m$. Thus, $[T(v_j)]_\gamma = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$.

   Now $[T]_\beta^\gamma = [[T(v_1)]_\gamma [T(v_2)]_\gamma \ldots [T(v_j)]_\gamma \ldots [T(v_n)]_\gamma] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$.

   We will represent this as $[T]_\beta^\gamma = (a_{ij})$. For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $U(v_j) = b_{1j}w_1 + b_{2j}w_2 + \ldots + b_{ij}w_i + \ldots + b_{mj}w_m$.

   Thus, $[T(v_j)]_\gamma = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$.

   Now $[T]_\beta^\gamma = [[U(v_1)]_\gamma [U(v_2)]_\gamma \ldots [U(v_j)]_\gamma \ldots [U(v_n)]_\gamma] = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{ij} & \ldots & b_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mj} & \ldots & b_{mn} \end{bmatrix}$.

   We will represent this as $[U]_\beta^\gamma = (b_{ij})$. Now $[T]_\beta^\gamma + [U]_\beta^\gamma = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ For each j, $1 \leq j \leq n$, $T(v_j) \in W$. Since γ is a basis for W, $\exists a_ij \in F$, $1 \leq i \leq m$ such that $(T+U)(v_j) = T(v_j) + U(v_j) = (a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m) + (b_{1j}w_1 + b_{2j}w_2 + \ldots + b_{ij}w_i + \ldots + b_{mj}w_m = (a_{1j}+b_{1j})w_1 + (a_{2j}+b_{2j})w_2 + \ldots + (a_{ij}+b_{ij})w_i + \ldots + (a_{mj}+b_{mj})w_m$ and $[T + U]_\beta^\gamma = (a_{ij} + b_{ij})$.

   (b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma \; \forall a \in F$. Let $a \in F$. From the part (a), $a[T]_\beta^\gamma = a(a_{ij}) = (aa_{ij}) = [aT]_\beta^\gamma$.

3. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \longrightarrow F^n$ by $T(x) = [x]_\beta$. Prove that T is linear. Please see your notes.

4. (Theorem 2.9 (page 86)) Let V, W, and Z be vector spaces over the field F, and let $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ be linear. Then $UT : V \longrightarrow Z$ is linear.
   Let $c \in F$, and $x, y \in V$.
   Then $(UT)(cx+y) = U(T(cx+y))$(definition of the composite UT) $= U(cT(x)+T(y))$(T is linear) $= cU(T(x)) + U(T(y)))$ ($U : W \longrightarrow Z$ is linear and $T(x), T(y) \in W$) $= c(UT)(x) + (UT)(y)$ (definition of the composite UT)

5. (Theorem 2.17 (page 100)) Let V and W be vector spaces over the field F, and let $T : V \longrightarrow W$ be invertible and linear. Then $T^{-1} : W \longrightarrow V$ is linear.
   Please see your notes.

6. (Theorem 2.5 (page 71)) Let V and W be vector spaces of equal (finite) dimension and $T : V \longrightarrow W$ be linear. Then the following conditions are equivalent.
   (a) T is 1-1.
   (b) T is onto.
   (c) rank(T) = dim(V).
   We will prove that (1): (a) $\Longrightarrow$ (b), (2): (b) $\Longrightarrow$ (c), (3): (c) $\Longrightarrow$ (a).
   Proof of (1). Assume T is 1-1. By Theorem 2.4 (page 71) $N(T) = \{0_V\}$. So, nullity(T)=0. By Theorem 2.2, nullity(T)+rank(T)=dim(V). Then rank(T)=dim(V). But dim(v)=dim(W). So rank(T)=dim(W). Now R(T) is a subspace of W and dim(R(T))= rank(T)=dim(W). Hence R(T)=W and T is onto.
   Proof of (2). Assume T is onto. Then R(T)=W (definition). Thus rank(T) = dim(W). But dim(v)=dim(W). So rank(T)=dim(V). Proof of (3). Assume rank(T)=dim(V). By Theorem 2.3, nullity(T)+rank(T)=dim(V). Then nullity(T)=0. So $N(T) = \{0_v\}$. By Theorem 2.4 (page 71), T is 1-1.

7. Let V and W be vector spaces of equal (finite) dimension and $T : V \longrightarrow W$ be linear. Prove that T is invertible if and only if rank(T) = dim(V).

8. (Lemma (page 101)) Let V and W be vector spaces of equal (finite) dimension and $T : V \longrightarrow W$ be linear. Then
   (a) V is finite-dimensional if and only if W is finite-dimensional.
   Proof not needed for Quiz # 3
   (b) If V and W are finite-dimensional, then dim(V) = dim(W).
   Proof not needed for Quiz # 3

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9. (Theorem 2.19 (page 103)) Let V and W be finite-dimensional vector spaces over the field F. Then V is isomorphic to W if and only if dim(V) = dim(W).
   Proof not needed for Quiz # 3.

10. For each of the following linear transformations T, determine whether T is invertible and justify your answer. It T is invertible, then find a formula for T.
    (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
    (b) $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(f(x)) = xf'(x)$.
    (c) $T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$ defined by $T(f(x)) = (f(0), f(1), f(2))$.
    (d) $T : V \longrightarrow V$, dim(V)=4, and with respect to the basis $\{v_1, v_2, v_3, v_4\}$, T is given by $T(v_1) = v_2$, $T(v_2) = v_1$, $T(v_3) = v_4$, $T(v_4) = v_3$.

12(a) Problem 2(a)/Exercises page 106. Solved above.
12(b) N(T): $T(a+bx+cx2+dx3) = (a+bx+cx2+dx3)' = b+2cx+3dx2$. $T(a+bx+cx2+dx3) = 0 \iff b+2cx+3dx2 = 0 \iff b = 0, c = 0, d = 0$. Thus $N(T) = \{a+bx+cx2+dx3 \in P_3(\mathbb{R}) | b = 0, c = 0, d = 0\} = \{a | a \in \mathbb{R}\} = \mathbb{R}$. By Theorem 2.4 (page 71), since $N(T) \neq \{0\}$, T is not 1-1. Hence, T is not invertible.
12(c) $f(x) = a + bx + cx2$. $f(0) = a, f(1) = a + b + c, f(2) = a + 2b + 4c$.
$T(f(x)) = (0,0,0) \iff (f(0), f(1), f(2)) = (0,0,0) \iff (a, a+b+c, a+2b+4c) = (0,0,0) \iff a = 0, a+b+c = 0, a+2b+4c = 0 \iff a = 0, b = 0, c = 0$. Thus, $N(T) = \{0\}$. $\iff a+b = 0, a = 0, c = 0, c+d = 0 \iff a = 0, b = 0, c = 0, d = 0$. Thus, $N(T) = \{0\}$. By Theorem 2.4 (page 71), T is 1-1. By Theorem 2.5 (page 71), T is onto. Hence T is invertible.
Formula for T^{-1}:
$T^{-1}(a,b,c) = a + (\frac{-3}{2}a + 2b - \frac{1}{2}c)x + (\frac{1}{2}a - b + \frac{1}{2}c)x2$.
12(d) R(T) = span$\{T(v_1), T(v_2), T(v_3), T(v_4)\}$ = span$\{v_2, v_1, v_4, v_3\}$= V. Thus, T is onto, by the definition of an onto mapping. By Theorem 2.5 (page 71), T is also onto. Hence T is invertible.
Formula for T^{-1}: $T^{-1}(v_1) = v_2$,$T^{-1}(v_2) = v_1$, $T^{-1}(v_3) = v_4$, $T^{-1}(v_4) = v_3$.
Note $T = T^{-1}$.

11. Which of the following pairs of vector spaces are isomorphic? Justify your answer.

(a) $V = \mathbb{R}^4$, $W = \{f(x) \in P_4(\mathbb{R}) | f(0) = 0\}$.
(b) $V = P_5(\mathbb{R})$, W= $M_{2\times3}(\mathbb{R})$.
(c) $V = \{A \in M_{2\times2}(\mathbb{R}) | tr(A) = 0\}$, $W = \mathbb{R}^4$.

13(a) Note $W = \{ax + bx2 + cx3 + dx4 | a, b, c, d \in \mathbb{R}\}$ = span$\{x, x2, x3, x4\}$.
Now dim(V) = dim($\mathbb{R}^4$) = 4 = dim(W). Hence, By Theorem 2.19 (page 103), V and W are isomorphic.
13(b) dim(V) = dim($P_5(\mathbb{R})$ = 5 =dim($M_{2\times3}(\mathbb{R})$ = dim(W). By Theorem 2.19, (page 103), V and W are isomorphic.
13(c) See 10 above / Exercises (page 106)