Math 252 Test # 2 Material Winter 2011 (Test Date: March 4, 2011)

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1. Use a triple integral in cylindrical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
   Ans. $2\int_0^{2\pi}[\int_0^a(\int_0^{\sqrt{a^2-r^2}} dz)rdr]d\theta = \frac{4}{3}\pi a^3$

2. Use cylindrical coordinates to solve the following problems.

   (a) Find the volume of the solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane. Ans. $\int_0^{2\pi}[\int_0^1(\int_0^{1-r^2} dz)rdr]d\theta = \frac{\pi}{2}$

   (b) A cylindrical hole of radius a is bored through the center of a solid sphere of radius $2a$. Find the volume of the hole. Ans. $2\int_0^{2\pi}[\int_0^{2a}(\int_0^{\sqrt{4a^2-r^2}} dz)rdr]d\theta = 4\sqrt{3}\pi a^3$

   (c) Find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$. Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{2\cos\theta}(\int_{r^2}^{2r\cos\theta} dz)rdr]d\theta = \frac{\pi}{2}$

   (d) Find the volume of the region bounded above by the plane $z = x$ and below by the paraboloid $z = x^2 + y^2$.
   Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{\cos\theta}(\int_{r^2}^{r\cos\theta} dz)rdr]d\theta = \frac{\pi}{32}$

   (e) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$ $(a > 0)$.
   Ans. $\int_0^{\pi}[\int_0^a(\int_{\frac{r^2}{a}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7)$

   (f) Find the volume of the region inside the cylinder $r = a\sin\theta$ which is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ where $b < a$.
   Ans. $\int_0^{\pi}[\int_0^{a\sin\theta}(\int_{b\sqrt{1-\frac{r^2}{a^2}}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7) = \frac{1}{9}a^2(a-b)(3\pi - 4)$

3. Evaluate $\int\int_R dA$, where R is the trapezoidal region with vertices given by $(0, 0), (5, 0), (\frac{5}{2}, \frac{5}{2})$, and $(\frac{5}{2}, -\frac{5}{2})$, by making the change of variables $x = 2u + 3v$ and $y = 2u - 3v$.
   Note $u = \frac{x+y}{4}$ and $u = \frac{x-y}{6}$. The region in the uv-plane is a square with vertices $(0, 0)$, $(\frac{5}{4}, 0)$, $(\frac{5}{4}, \frac{5}{6})$, and $(0, \frac{5}{6})$. Ans. $\frac{5}{4} \times \frac{5}{6} = \frac{25}{24}$.

4. Let R be the region bounded by the line $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$. Evaluate $\int\int_R 3xydxdy$ by making the change of variables
   $x = \frac{1}{3}(2u + v)$, $y = \frac{1}{3}(u - v)$.
   We note that $J = \frac{1}{3}$ and $u = x + y$ and $v = x - 2y$. From the given equations, it follows that $1 \leq u(= x + y) \leq 4$ and $-4 \leq v\ (= x - 2y)\ \leq 0$. Thus $\int\int_R 3xydxdy = \int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$.
   Alternatively, the given region is a rectangle with vertices $(-\frac{2}{3}, \frac{5}{3})$,$(\frac{2}{3}, \frac{1}{3})$, $(\frac{8}{3}, \frac{4}{3})$, and $(\frac{4}{3}, \frac{8}{3})$ and the corresponding region in the uv-plane is the rectangle with vertices $(1, 0), (4, 0), (1, -4)$, and $(4, -4) = [1, 4] \times [-4, 0]$. Ans. $\int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$

5. Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral $\int\int_R(x+y)^2\sin^2(x-y)dxdy$ by making the change of variables $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. The corresponding region in the uv-plane is the square with vertices $(1, 1), (3, 1), (1, -1)$, and $(3, -1)$. $J = \frac{1}{2}$
   $\int\int_R(x + y)^2\sin^2(x - y)dxdy = \int_{-1}^{1}[\int_1^3 u^2\sin^2 v\frac{1}{2}du]dv = \frac{13}{6}(2 - \sin 2)$

6. Let R be the region bounded by the parallelogram with vertices $(0,0)$, $(4,0)$, $(3,3)$, and $(7,3)$. Evaluate the integral $\int\int_R y(x-y)dxdy$ by making the change of variables $x = u+v$, $y = u$.
   Ans. 36

7. Let B be the square with vertices at $(0,1)$, $(1,0)$, $(2,1)$, and $(1,2)$.
   Evaluate $\int\int_B 60\,x\,y\,dx\,dy$
   by making the change of variables $x = \frac{1}{2}(u+v)$, $y = -\frac{1}{2}(u-v)$.
   Ans. 120

8. Use a triple integral in spherical coordinates to find the volume of the sphere $x^2+y^2+z^2 = a^2$.
   Ans. $\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi d\theta = \frac{4}{3}\pi a^3$.

9. Use a triple integral in spherical coordinates to find the volume of the sphere $x^2+y^2+z^2 = 2z$.
   Ans. $\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi d\theta = \frac{4}{3}\pi$.

10. Find the volume of the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.
    Ans. $\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi d\theta = \frac{4}{3}\pi a^3(1-\cos\alpha)$.

11. If $0 < b < a$ and $0 < \alpha < \pi$, find the volume of the region bounded by the concentric spheres $\rho = b$, $\rho = a$ and the cone $\phi = \alpha$.
    Ans. $\int_0^{2\pi}\int_0^{\alpha}\int_b^{a} \rho^2 \sin\phi\, d\rho\, d\phi d\theta = \frac{2\pi}{3}(a^3-b^3)(1-\cos\alpha)$.

12. Find the volume of the solid region bounded below by the upper nappe of the cone $z^2 = x^2+y^2$ and above by the sphere $x^2+y^2+z^2 = 9$.
    Ans. $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi\, d\rho\, d\phi d\theta = 9\pi(2-\sqrt{2})$.

13. Evaluate $\int\int\int_E 16zdxdy$, where E is the upper half of the sphere $x^2+y^2+z^2 = 1$.
    Ans. $\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} (16\rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi d\theta = 4\pi$.

14. Convert $\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dzdxdy$ into an integral in

    (a) spherical coordinates. Ans. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{3\sqrt{2}} (\rho^2)\rho^2 \sin\phi\, d\rho\, d\phi d\theta$.

    (b) cylindrical coordinates. $\int_0^{\frac{\pi}{2}}[\int_0^3(\int_r^{\sqrt{18-r^2}}(r^2+z^2)\,dz)r\,dr]\,d\theta$

15. Convert $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_a^{a+\sqrt{a^2-x^2-y^2}} xdzdydx$
    into an integral in

    (a) spherical coordinates. Ans. $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{a\sec phi}^{2a\cos\phi} \rho^3 \sin^2\phi\, \cos\,\theta d\rho\, d\phi d\theta$.

    (b) cylindrical coordinates. Ans. $\int_0^{2\pi}\int_0^{a}(\int_a^{a+\sqrt{a^2-r^2}}(rcos\theta)\,dz)r\,dr\,d\theta$.

16. Evaluate $\int_C (x-3y)ds$, where C is the line segment from $(0,0)$ to $(1,2)$ and C is parameterized as:

    (a) $x = t$, $y = 2t$, $0 \le t \le 1$.

    (b) $x = \sin t$, $y = 2\sin t$, $0 \le t \le \frac{\pi}{2}$.

17. Evaluate $\int_C (x^2 - y + 3z)ds$, where C is the line segment from $(0,0,0)$ to $(1,2,1)$. [Answer. $\int_0^1 (t^2 - 2t + 3t)\sqrt{6}dt = \frac{5\sqrt{6}}{6}$].

18. Evaluate $\int_C xds$, where C is the line segment from $(0,0)$ to $(1,1)$ and $y = x^2$ from $(1,1)$ to $(0,0)$. [Answer. $\int_0^1 t\sqrt{2}dt + \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{\frac{3}{2}} - 1)$].

19. Evaluate the line integrals $\int_C (x-3y)dx$, and $\int_C (x-3y)dy$, if C is the part of the parabola $x = y^2$ that joins the points $(1,1)$ and $(4,2)$ [Answer. $\int_1^2 (t^2-3t)2tdt = \frac{-13}{2}$ and $\int_1^2 (t^2-3t)dt = \frac{-13}{6}$].

20. Evaluate the line integral $\int_C (y^2dx - x^2dy)$ along the two curves given below:
    (a) C_1: The parabola $x = t$, $y = t^2$ joining the two points $(0,0)$ and $(2,4)$. [Answer. $\int_0^2 (t^4 - 2t^3)dt = \frac{-8}{5}$].
    (b) C_2: The line $x = t$, $y = 2t$ joining the two points $(0,0)$ and $(2,4)$. [Answer. $\int_0^2 2t^2dt = \frac{16}{3}$].
21. Evaluate the line integral $\int_C xy^2dx - (x+y)dy$, where C is
    (a) The straight line segment from $(0,0)$ and $(1,2)$. [Answer. $\int_0^1 (4t^3 - 6t)dt = -2$].
    (b) The parabolic path from $(0,0)$ and $(2,4)$. [Answer. $\int_0^1 (4t^5 - 4t^2 - 8t^3)dt = \frac{-8}{3}$].
    (c) The broken line from $(0,0)$ to $(1,0)$ to $(1,2)$. [Answer. $-\int_0^2 (1+t)dt = -4$].
22. Evaluate the line integral $\int_C xy^2dx - (x+y)dy$, where C is the broken line joining the points $(0,0)$, $(1,1)$, $(2,1)$ in this order.
23. Evaluate the line integral $\int_C \frac{dx}{y} + \frac{dy}{x}$, where C is the part of the hyperbola $xy = 4$ from $(1,4)$ to $(4,1)$. [Answer. $\int_1^4 (\frac{t}{4} - \frac{4}{t^3})dt = 0$].
24. Evaluate the line integral $\int_C xdx + x^2dy$ from $(-1,0)$ to $(1,0)$.
    (a) Along the x-axis. [Answer. $\int_{-1}^1 tdt = 0$].
    (b) Along the semicircle $y = \sqrt{1-x^2}$ [Answer. $\int_0^\pi (-\sin t + 1 - \sin^t)dt = 0$].
    (c) Along the broken line from $(-1,0)$ to $(0,1)$ to $(1,1)$. [Answer. $\int_{-1}^0 (t+t^2)dt + \int_0^1 tdt - \int_0^1 dt = -\frac{1}{6} + \frac{1}{2} - 1 = -\frac{2}{3}$].
25. Evaluate the line integral $\int_C ydx + (x+2y)dy$ from $(1,0)$ to $(0,1)$, where C is
    (a) the arc of the circle $x = \cos t$, $y = \sin t$; [Answer. $\int_0^{\frac{\pi}{2}} (\cos^t - \sin^2 t + 2\sin t \cos t)dt = 1$].
    (b) the straight line segment $y = 1 - x$; [Answer. $\int_1^0 (-1)dx = 1$].
    (c) the broken line from $(1,0)$ to $(1,1)$ to $(0,1)$. [Answer. $\int_0^1 (1+2y)dy + \int_1^0 dx = 1$].
26. Evaluate $\int_C (3x+4y)dx + (2x+3y^2)dy$, where C is the circle $x^2+y^2 = 4$ traversed counterclockwise from $(2,0)$. [Answer. $x = 2\cos t, y = 2\sin t,\ 0 \le t \le 2\pi$, $\int_0^{2\pi} ((24\sin^2 t - 12\sin t)\cos t + (8 - 24\sin^2 t))dt = -8\pi$].
27. Evaluate $\int_C (x+2)ds$, where C is the curve represented by $\mathbf{c}(t) = t\mathbf{i} + \frac{4}{3}t^{\frac{3}{2}}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$, $0 \le t \le 6\pi$.
28. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, if $\mathbf{F} = xy^2\mathbf{i} + x^2z\mathbf{j} - (y-x)\mathbf{k}$, C is the curve $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$.
29. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, if $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} - (y-x)\mathbf{k}$, C is the straight line segment from thee point (1,0,-1) to the point(3,3,5).
30. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, if $\mathbf{F}(x,y) = (x+y)\mathbf{i} + (y^2-x)\mathbf{j}$, where C is the closed curve that begins at $(1,0)$, proceeds along the upper half of the unit circle to $(-1,0)$, and returns to $(1,0)$ along the x-axis.
31. (a) Evaluate $\int_S xy^4ds$, where C is the right half of the circle, $x^2+y^2 = 1$. [Answer. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4(4)dt = \frac{8192}{5}$].
    (b) Evaluate $\int_S 4x^3ds$, were C is the line segment from (-2,-1) to (1,2). [Answer. $\int_{-2}^0 4(-2+3t)^3\sqrt{18}dt = -15\sqrt{2}$].
    (c) Evaluate $\int_S 4x^3ds$, were C is the line segment from (1,2) to (-2,-1). [Answer. $\int_0^1 4(1-3t)^3\sqrt{18}dt = -15\sqrt{2}$].

32. Evaluate $\int_C x ds$ for each of the following curves. (a) C_1: $y = x^2$, $-1 \leq x \leq 1$ (b) C_2: The line segment from(-1,1) to (1,1). (c) C_3: The line segment from(1,1) to (-1,1). [Answer: 0 for each part (a)-(c)].

33. Evaluate $\int_C xyz ds$, where C is the helix given by, $x = \cos t$, $y = \sin t$, $z = 3t$, $0 \leq t \leq 4\pi$.

34. Evaluate $\int_C \sin(\pi y)dy + yx^2dx$, where C is the line segment from (0,2) to (1,4). [Answer. $\frac{7}{6}$].

35. Evaluate $\int_C \sin(\pi y)dy + yx^2dx$, where C is the line segment from (1,4) to (0,2). [Answer. $-\frac{7}{6}$].

36. Evaluate $\int_C ydx + xdy + zdx$, where C is given by, $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \leq t \leq 2\pi$. [Answer. $\int_0^{2\pi}(-\sin^2 t + \cos^2 t + 2t^3)dt = 8\pi^4$].

37. Evaluate $\int_C(2xydx + x^2dy)$ if:
    (a) C consists of the line segments from $(3, 1)$ to $(5, 1)$ and from $(5, 1)$ to $(5, 6)$.
    (b) C is the line segments from $(3, 1)$ to $(5, 6)$.
    (c) C is the part of the parabola $x = 2t + 1$, $y = 2t^2 - t$, $1 \leq t \leq 2$. [Answer: 141, for (a), (b), (c)]

38. Find a potential function for
    (a) $\mathbf{F}(x, y) = (2xy + 24x)\mathbf{i} + (x^2 + 16y)\mathbf{j}$. (Answer: $x^2y + 12x^3 + 16y + C$)
    (b) $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$. (Answer: $x^2y - \frac{y^2}{2} + C$)
    (c) $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$. (Answer: $x^2y + z^2y + C$)
    (d) $\mathbf{F}(x, y, z) = (y \cos x + 2xe^y)\mathbf{i} + (\sin x + x^2e^y + 4)\mathbf{j}$. (Answer: $y \sin x x^2e^y + 4y + C$)
    (e) $\mathbf{F}(x, y) = (x^2 - yx)\mathbf{i} + (y^2 + xy)\mathbf{j}$.
    (f) $\mathbf{F}(x, y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + 2y)\mathbf{j}$. (Answer: $x^2e^{xy} + y^2 + C$)
    (g) $\mathbf{F}(x, y, z) = 2xy^3z^4\mathbf{i} + (3x^2y^2z^4)\mathbf{j} + 4x^2y^3z^3\mathbf{k}$. (Answer: $x^2y^3z^4$)
    (h) $\mathbf{F}(x, y, z) = 2x \cos y - 2z^3\mathbf{i} + (3 + 2ye^z - x^2 \sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$. (Answer: $y^2e^z - 2xz^3 + x^2 \cos y + 3y + C$)

39. Evaluate $\int_C \mathbf{F}.d\mathbf{r}$, where C is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$. (Answer: $x^2y - \frac{y^2}{2} + C$, 4)

40. Evaluate $\int_C \mathbf{F}.d\mathbf{r}$, where C is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 2, 3)$ and $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$. (Answer: $x^2y + z^2y + C$, 17)

41. Use Green's theorem to evaluate the line integral $\oint_C[xydx + x^2y^3dy]$, where C is the triangle with vertices $(0, 0), (1, 0)$, and $(1, 2)$.
    (Answer: $P = xy$, $Q = x^2y^3$, the region enclosed is $0 \leq x \leq 1, 0 \leq y \leq 2x$, $\int_0^1 \int_0^{2x}(Q_x - P_y)dydx = \int_0^1 \int_0^{2x}(2xy^3 - x)dydx = \frac{2}{3}$).

42. Use Green's theorem to evaluate the line integral $\oint_C[y^3dx + x^3dy]$, where C is the circle of radius 2 centered at $(0, 0)$.
    (Answer: $P = y^3$, $Q = x^3$, the region enclosed is $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$, $\int_0^{2\pi} \int_0^2(Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2(3r^2 \cos^2 \theta - 3r^2 \sin^2 \theta)rdrd\theta = 0$).

43. Use Green's theorem to evaluate the line integral $\oint_C[(-2xy+y^2)dx+x^2dy]$, where C is the boundary of the region R enclosed by $y = 4x$ and $y = 2x^2$.
    (Answer: $P = -2yx + y^2$, $Q = x^2$, the region enclosed is $0 \leq x \leq 2, 2x^2 \leq y \leq 4x$, $\int_0^2 \int_{2x^2}^{4x}(Q_x - P_y)dydx = \int_0^2 \int_{2x^2}^{4x}(4x - 2y)dydx = \frac{-32}{5}$).

44. Evaluate the line integral $\oint_C (3x - y)dx + (x + 5y)dy$ around the unit circle $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.
(Answer: $P = 3x - y$, $Q = x + 5y$, the region enclosed is $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$, $\int_0^{2\pi} \int_0^1 (Q_x - P_y) r dr d\theta = \int_0^{2\pi} \int_0^1 (1 - (-1)) r dr d\theta = 2\pi$).

45. Evaluate the line integral $\oint_C [(e^{-x^2} + y^2)dx + (\ln y - x^2))dy]$, where C is the square with vertices $(0,0), (1,0), (1,1), (0,1)$.
(Answer: $P = e^{-x^2} + y^2$, $Q = \ln y - x^2$, the region enclosed is $0 \leq x \leq 1, 0 \leq y \leq 1$, $\int_0^1 \int_0^1 (Q_x - P_y) dy dx = \int_0^1 \int_0^1 (-2x - 2y) dy dx = -2$).

46. Evaluate the line integral $\oint_C (2y + \sqrt{1 + x^5})dx + (5x - e^{y^2})dy$ around the circle $x^2 + y^2 = 4$.
(Answer: $P = 2y + \sqrt{1 + x^5}$, $Q = 5x - e^{y^2}$, the region enclosed is $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$, $\int_0^{2\pi} \int_0^2 (Q_x - P_y) r dr d\theta = \int_0^{2\pi} \int_0^2 (5 - 2) r dr d\theta = 12\pi$).

47. If R is any region to which Green's Theorem is applicable, show that the area of R is given by the formula
$A = \frac{1}{2} \oint_C -y dx + x dy$, or
$A = \oint_C x dy$, or
$A = \oint_C (-y) dx$.

48. Use Green's theorem to find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(Answer. Parametric equations for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: $x = a \cos t$, $y = b \sin t$, $0 \leq t \leq 2\pi$. Area $= \frac{1}{2} \oint_C -y dx + x dy = \frac{1}{2} \int_0^{2\pi} [(-b \sin t)(-a \sin t dt) + (a \cos t)(b \cos t dt)] = \pi ab$.)

49. Use Green's theorem to find the area of the region enclosed by the circle $x^2 + y^2 = a^2$.
(Answer. Parametric equations for the circle $x^2 + y^2 = a^2$: $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$. Area $= \frac{1}{2} \oint_C -y dx + x dy = \frac{1}{2} \int_0^{2\pi} [(-a \sin t)(-a \sin t dt) + (a \cos t)(a \cos t dt)] = \pi a^2$.)

50. Use Green's theorem to find the area of the region enclosed by $y = x^2$ and $y = x + 2$.
(Answer. Parametric equations for the boundary of the region: $C_1 : x = t$, $y = t^2$, $-1 \leq t \leq 2$; $C_2 : x = -3t + 2$, $y = -3t + 4$ Area $= \frac{1}{2} \oint_C -y dx + x dy = \frac{1}{2} \int_{-1}^{2} [(-t^2)(dt) + (t)(2t dt)] + \frac{1}{2} \int_0^1 [(3t - 4)(-3dt) + (-3t + 2)(-3dt)] \frac{9}{2}$.)

51. Use Green's theorem to find the area of the region enclosed by $y = x^2 - 1$ and $y = 0$.
(Answer. Parametric equations for the boundary of the region: $C_1 : x = t$, $y = 1 - t^2$, $-1 \leq t \leq 1$, $C_2 : x = -t$, $y = 0$, $-1 \leq t \leq 1$, Area $= \frac{1}{2} \oint_{C_1} -y dx + x dy + \frac{1}{2} \oint_{C_2} -y dx + x dy = \frac{1}{2} \int_{-1}^{1} [(-(1 - t^2)(dt) + (t)(-2t dt)] + \frac{1}{2} \int_{-1}^{1} [(0)(dt) + (-t)(0dt)]$.)

52. Use Green's theorem to find the area under one arc of the cycloid $x = 2\pi t - \sin 2\pi t$ and $y = 1 - \cos 2\pi t$; $0 \leq t \leq 1$.
(Answer. Parametric equations for the boundary of the region: $C_1 : x = t$, $y = 0$, $0 \leq t \leq 2\pi$, $C_2 : x = 2\pi t - \sin 2\pi t$, $y = 1 - \cos 2\pi t$, $0 \leq t \leq 1$, Area $= \frac{1}{2} \oint_{C_1} -y dx + x dy + \frac{1}{2} \oint_{C_2} -y dx + x dy = \frac{1}{2} \int_0^1 [(-(0)(dt) + (t)(0dt)] - \frac{1}{2} \int_0^1 [(-(2\pi t - \sin 2\pi t)((2\pi - 2\pi \cos 2\pi t)dt) + (2\pi t - \sin 2\pi t)(2\pi \sin 2\pi t dt)] = 3\pi$.)

53. Determine the surface given by the parametric equations $\mathbf{r}(u, v) = u\mathbf{i} + u \cos v\mathbf{j} + u \sin v\mathbf{k}$.

54. Give parametric representations for each of the following surfaces.

    (a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$.
    $x = 5y^2 + 2z^2 - 10$, $y = y$, $z = z$

    (b) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ that is in front of the yz-plane.
    $x = 5y^2 + 2z^2 - 10$, $y = y$, $z = z$, $x \geq 0$.

(c) The sphere $x^2 + y^2 + 2z^2 = 30$.
$x = \sqrt{30} \sin\phi \cos\theta$, $y = \sqrt{30} \sin\phi \sin\theta$, $z = \sqrt{30} \cos\phi$; $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

(d) The cylinder $y^2 + 2z^2 = 25$.
$x = x$, $y = 5 \sin\theta$, $z = 5 \cos\theta$, $0 \leq \theta \leq 2\pi$.