

EXERCISES Page 528 1, 2, 3, 5, 13 EXERCISES Page 558 2, 3, 4, 7, 9, 13, 14

1. Find a potential function for

(a)  $\mathbf{F}(x, y) = (2xy + 24x)\mathbf{i} + (x^2 + 16y)\mathbf{j}$ . (Answer:  $x^2y + 12x^3 + 16y + C$ )

(b)  $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$ . (Answer:  $x^2y - \frac{y^2}{2} + C$ )

(c)  $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$ . (Answer:  $x^2y + z^2y + C$ )

(d)  $\mathbf{F}(x, y, z) = (y \cos x + 2xe^y)\mathbf{i} + (\sin x + x^2e^y + 4)\mathbf{j}$ . (Answer:  $y \sin xx^2e^y + 4y + C$ )

(e)  $\mathbf{F}(x, y) = (x^2 - yx)\mathbf{i} + (y^2 + xy)\mathbf{j}$ .

(f)  $\mathbf{F}(x, y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + 2y)\mathbf{j}$ . (Answer:  $x^2e^{xy} + y^2 + C$ )

(g)  $\mathbf{F}(x, y, z) = 2xy^3z^4\mathbf{i} + (3x^2y^2z^4)\mathbf{j} + 4x^2y^3z^3\mathbf{k}$ . (Answer:  $x^2y^3z^4$ )

(h)  $\mathbf{F}(x, y, z) = 2x \cos y - 2z^3\mathbf{i} + (3 + 2ye^z - x^2 \sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$ . (Answer:  $y^2e^z - 2xz^3 + x^2 \cos y + 3y + C$ )

2. Use Green's theorem to evaluate the line integral  $\oint_C [xydx + x^2y^3dy]$ , where  $C$  is the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 2)$ .

(Answer:  $P = xy$ ,  $Q = x^2y^3$ , the region enclosed is  $0 \leq x \leq 1, 0 \leq y \leq 2x$ ,  $\int_0^1 \int_0^{2x} (Q_x - P_y)dydx = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \frac{2}{3}$ ).

3. Use Green's theorem to evaluate the line integral  $\oint_C [y^3dx + x^3dy]$ , where  $C$  is the circle of radius 2 centered at  $(0, 0)$ .

(Answer:  $P = y^3$ ,  $Q = x^3$ , the region enclosed is  $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$ ,  $\int_0^{2\pi} \int_0^2 (Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2 (3r^2 \cos^2 \theta - 3r^2 \sin^2 \theta)rdrd\theta = 0$ ).

4. Use Green's theorem to evaluate the line integral  $\oint_C [(-2xy + y^2)dx + x^2dy]$ , where  $C$  is the boundary of the region  $R$  enclosed by  $y = 4x$  and  $y = 2x^2$ .

(Answer:  $P = -2yx + y^2$ ,  $Q = x^2$ , the region enclosed is  $0 \leq x \leq 2, 2x^2 \leq y \leq 4x$ ,  $\int_0^2 \int_{2x^2}^{4x} (Q_x - P_y)dydx = \int_0^2 \int_{2x^2}^{4x} (4x - 2y)dydx = \frac{-32}{5}$ ).

5. Evaluate the line integral  $\oint_C (3x - y)dx + (x + 5y)dy$  around the unit circle  $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq 2\pi$ .

(Answer:  $P = 3x - y$ ,  $Q = x + 5y$ , the region enclosed is  $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$ ,  $\int_0^{2\pi} \int_0^1 (Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^1 (1 - (-1))rdrd\theta = 2\pi$ ).

6. Evaluate the line integral  $\oint_C [(e^{-x^2} + y^2)dx + (\ln y - x^2)dy]$ , where  $C$  is the square with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(0, 1)$ .

(Answer:  $P = e^{-x^2} + y^2$ ,  $Q = \ln y - x^2$ , the region enclosed is  $0 \leq x \leq 1, 0 \leq y \leq 1$ ,  $\int_0^1 \int_0^1 (Q_x - P_y)dydx = \int_0^1 \int_0^1 (-2x - 2y)dydx = -2$ ).

7. Evaluate the line integral  $\oint_C (2y + \sqrt{1 + x^5})dx + (5x - e^{y^2})dy$  around the circle  $x^2 + y^2 = 4$ .

(Answer:  $P = 2y + \sqrt{1 + x^5}$ ,  $Q = 5x - e^{y^2}$ , the region enclosed is  $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$ ,  $\int_0^{2\pi} \int_0^2 (Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2 (5 - 2)rdrd\theta = 12\pi$ ).

8. If  $R$  is any region to which Green's Theorem is applicable, show that the area of  $R$  is given by the formula

$A = \frac{1}{2} \oint_C -ydx + xdy$ , or

$A = \oint_C xdy$ , or

$A = \oint_C (-y)dx$ .

9. Use Green's theorem to find the area of the region enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .  
 (Answer. Parametric equations for the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ :  $x = a \cos t$ ,  $y = b \sin t$ ,  $0 \leq t \leq 2\pi$ .  
 Area =  $\frac{1}{2} \oint_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} [(-b \sin t)(-a \sin t dt) + (a \cos t)(b \cos t dt)] = \pi ab$ .)
10. Use Green's theorem to find the area of the region enclosed by the circle  $x^2 + y^2 = a^2$ .  
 (Answer. Parametric equations for the circle  $x^2 + y^2 = a^2$ :  $x = a \cos t$ ,  $y = a \sin t$ ,  $0 \leq t \leq 2\pi$ .  
 Area =  $\frac{1}{2} \oint_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} [(-a \sin t)(-a \sin t dt) + (a \cos t)(a \cos t dt)] = \pi a^2$ .)
11. Use Green's theorem to find the area of the region enclosed by  $y = x^2$  and  $y = x + 2$ .  
 (Answer. Parametric equations for the boundary of the region:  $C_1 : x = t$ ,  $y = t^2$ ,  $-1 \leq t \leq 2$ ;  
 $C_2 : x = -3t + 2$ ,  $y = -3t + 4$  Area =  $\frac{1}{2} \oint_C -ydx + xdy = \frac{1}{2} \int_{-1}^2 [(-t^2)(dt) + (t)(2t dt)] + \frac{1}{2} \int_0^1 [(3t - 4)(-3dt) + (-3t + 2)(-3dt)] \frac{9}{2}$ .)
12. Use Green's theorem to find the area of the region enclosed by  $y = x^2 - 1$  and  $y = 0$ .  
 (Answer. Parametric equations for the boundary of the region:  $C_1 : x = t$ ,  $y = 1 - t^2$ ,  $-1 \leq t \leq 1$ ,  
 $C_2 : x = -t$ ,  $y = 0$ ,  $-1 \leq t \leq 1$ , Area =  $\frac{1}{2} \oint_{C_1} -ydx + xdy + \frac{1}{2} \oint_{C_2} -ydx + xdy = \frac{1}{2} \int_{-1}^1 [(-(1 - t^2)(dt) + (t)(-2t dt)] + \frac{1}{2} \int_{-1}^1 [(0)(dt) + (-t)(0 dt)]$ .)
13. Use Green's theorem to find the area under one arc of the cycloid  $x = 2\pi t - \sin 2\pi t$  and  $y = 1 - \cos 2\pi t$ ;  $0 \leq t \leq 1$ .  
 (Answer. Parametric equations for the boundary of the region:  $C_1 : x = t$ ,  $y = 0$ ,  $0 \leq t \leq 2\pi$ ,  $C_2 : x = 2\pi t - \sin 2\pi t$ ,  $y = 1 - \cos 2\pi t$ ,  $0 \leq t \leq 1$ , Area =  $\frac{1}{2} \oint_{C_1} -ydx + xdy + \frac{1}{2} \oint_{C_2} -ydx + xdy = \frac{1}{2} \int_0^1 [(-(0)(dt) + (t)(0 dt)] - \frac{1}{2} \int_0^1 [(-(2\pi t - \sin 2\pi t)((2\pi - 2\pi \cos 2\pi t) dt) + (2\pi t - \sin 2\pi t)(2\pi \sin 2\pi t dt)] = 3\pi$ .)