Math 252 Sample Quiz #3 Winter 2011

EXERCISES Page 390 1-5, 8, 9, 13, 14, 17, 19, 21, 22, 23-25

1. Use a triple integral in cylindrical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
   Ans. $2\int_0^{2\pi}[\int_0^a(\int_0^{\sqrt{a^2-r^2}} dz)rdr]d\theta = \frac{4}{3}\pi a^3$

2. Use cylindrical coordinates to solve the following problems.
   (a) Find the volume of the solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane. Ans. $\int_0^{2\pi}[\int_0^1(\int_0^{1-r^2} dz)rdr]d\theta = \frac{\pi}{2}$
   (b) A cylindrical hole of radius a is bored through the center of a solid sphere of radius $2a$. Find the volume of the hole. Ans. $2\int_0^{2\pi}[\int_0^{2a}(\int_0^{\sqrt{4a^2-r^2}} dz)rdr]d\theta = 4\sqrt{3}\pi a^3$
   (c) Find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$. Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{2\cos\theta}(\int_{r^2}^{2r\cos\theta} dz)rdr]d\theta = \frac{\pi}{2}$
   (d) Find the volume of the region bounded above by the plane $z = x$ and below by the paraboloid $z = x^2 + y^2$.
   Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{\cos\theta}(\int_{r^2}^{r\cos\theta} dz)rdr]d\theta = \frac{\pi}{32}$
   (e) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$ $(a > 0)$.
   Ans. $\int_0^{\pi}[\int_0^a(\int_{\frac{r^2}{a}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7)$
   (f) Find the volume of the region inside the cylinder $r = a\sin\theta$ which is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ where $b < a$.
   Ans. $\int_0^{\pi}[\int_0^{a\sin\theta}(\int_{b\sqrt{1-\frac{r^2}{a^2}}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7) = \frac{1}{9}a^2(a - b)(3\pi - 4)$

3. Evaluate $\int\int_R dA$, where R is the trapezoidal region with vertices given by $(0, 0), (5, 0), (\frac{5}{2}, \frac{5}{2})$, and $(\frac{5}{2}, -\frac{5}{2})$, by making the change of variables $x = 2u + 3v$ and $y = 2u - 3v$.
   Note $u = \frac{x+y}{4}$ and $u = \frac{x-y}{6}$. The region in the uv-plane is a square with vertices $(0, 0)$, $(\frac{5}{4}, 0)$, $(\frac{5}{4}, \frac{5}{6})$, and $(0, \frac{5}{6})$. Ans. $\frac{5}{4} \times \frac{5}{6} = \frac{25}{24}$.

4. Let R be the region bounded by the line $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.
   Evaluate $\int\int_R 3xydxdy$ by making the change of variables
   $x = \frac{1}{3}(2u + v)$, $y = \frac{1}{3}(u - v)$.
   We note that $J = \frac{1}{3}$ and $u = x + y$ and $v = x - 2y$. From the given equations, it follows that $1 \le u(= x + y) \le 4$ and $-4 \le v\,(= x - 2y) \le 0$. Thus $\int\int_R 3xydxdy = \int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$.
   Alternatively, the given region is a rectangle with vertices $(-\frac{2}{3}, \frac{5}{3})$,$(\frac{2}{3}, \frac{1}{3})$, $(\frac{8}{3}, \frac{4}{3})$, and $(\frac{4}{3}, \frac{8}{3})$ and the corresponding region in the uv-plane is the rectangle with vertices $(1, 0), (4, 0), (1, -4)$, and $(4, -4) = [1, 4] \times [-4, 0]$. Ans. $\int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$

5. Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral $\int\int_R(x+y)^2\sin^2(x-y)dxdy$ by making the change of variables $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. The corresponding region in the uv-plane is the square with vertices $(1, 1), (3, 1), (1, -1)$, and $(3, -1)$. $J = \frac{1}{2}$
   $\int\int_R(x + y)^2\sin^2(x - y)dxdy = \int_{-1}^1[\int_1^3 u^2\sin^2 v\frac{1}{2}du]dv = \frac{13}{6}(2 - \sin 2)$

6. Let R be the region bounded by the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(3, 3)$, and $(7, 3)$. Evaluate the integral $\int\int_R y(x - y)dxdy$ by making the change of variables $x = u + v$, $y = u$.
   Ans. 36

7. Let B be the square with vertices at $(0,1)$, $(1,0)$, $(2,1)$, and $(1,2)$.
Evaluate $\int\int_B 60\, x\, y\, dx\, dy$
by making the change of variables $x = \frac{1}{2}(u+v)$, $y = -\frac{1}{2}(u-v)$.
Ans. 120

8. Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) dzdxdy$ into an integral in cylindrical coordinates.
$\int_0^{\frac{\pi}{2}} [\int_0^3 (\int_r^{\sqrt{18-r^2}} (r^2+z^2)\, dz) r\, dr]\, d\theta$

9. Convert $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_a^{a+\sqrt{a^2-x^2-y^2}} xdzdydx$
into an integral in cylindrical coordinates. Ans. $\int_0^{2\pi} \int_0^a (\int_a^{a+\sqrt{a^2-r^2}} (rcos\theta)\, dz) r\, dr\, d\theta$.