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EXERCISES
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1. Find the volume above the xy-plane bounded by the paraboloid $z = x^2+y^2$ and the planes $x = \pm 1$, $y = \pm 1$.
   Ans. $\int_{-1}^{1}[\int_{-1}^{1}(x^2 + y^2)\, dx]\, dy$

2. Find the volume above the xy-plane bounded by the cylinder $x^2+y^2 = 1$ and the plane $x+y+z = 2$.
   Ans. $\int_{-1}^{1}[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2 - x - y)\, dy]\, dx$

3. Find the volume above the xy-plane bounded by the cylinder $y = 4 - x^2$ and the planes $y = 3x$ and $z = x + 4$.
   Ans. $\int_{-4}^{1}[\int_{3x}^{4-x^2}(x + 4)\, dy]\, dx$

4. Find the volume of the solid bounded by the coordinate planes, the planes $x = 2$ and $y = 5$, and the surface $2z = xy$.
   Ans. $\int_{0}^{2}[\int_{0}^{5}(\frac{xy}{2})\, dy]\, dx$

5. Find the volume above the xy-plane bounded by the cylinder $x^2 + y^2 = 9$ and the paraboloid $3z = x^2 + y^2$.
   Ans. $\int_{-3}^{3}[\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}((\frac{x^2+y^2}{3}))\, dy]\, dx$

6. Find the volume of the solid in the first octant bounded by the cylinder $4y = x^2$ and the planes $x = 0$, $z = 0$, $y = 4$, and $x - y + 2z = 2$.
   Ans. $\int_{0}^{4}[\int_{\frac{x^2}{4}}^{4}(\frac{2-x-y}{2})\, dy]\, dx$

7. Change the order of integration of
   (a) $\int_{0}^{1}[\int_{y}^{1}\frac{1}{1+x^4}dx]dy$.
   Ans. $\int_{0}^{1}[\int_{0}^{x}\frac{1}{1+x^4}dy]dx$
   (b) $\int_{0}^{3}[\int_{x^2}^{2x+3}xdy]dx$.
   Ans. $\int_{0}^{3}[\int_{0}^{\sqrt{y}}xdx]dy + \int_{3}^{9}[\int_{\frac{y-3}{2}}^{\sqrt{y}}xdy]dx$

y - simple region:
The inside integral is from the bottom curve $y = g_1(x)$ to the top curve $y = g_2(x)$, where $a \le x \le b$.
x - simple region:
The inside integral is from the left curve $x = h_1(y)$ to the right curve $x = h_2(y)$, where $c \le y \le d$.
The area A of the region $R = \int\int_R dA = \int\int_R dxdy$

1. Sketch the region over which the integral extends, change the order of integration, evaluate.
   (a) $\int_0^1 [\int_y^1 \frac{1}{1+x^4} dx] dy$.
   Ans. $\int_0^1 [\int_0^x \frac{1}{1+x^4} dy] dx$
   (b) $\int_0^3 [\int_{x^2}^{2x+3} x dy] dx$.
   Ans. $\int_0^3 [\int_0^{\sqrt{y}} x dx] dy + \int_3^9 [\int_{\frac{y-3}{2}}^{\sqrt{y}} x dy] dx$
   (c) $\int_0^1 [\int_{x^2}^x (2x + 2y) dy] dx$.
   Ans. $\int_0^1 [\int_y^{\sqrt{y}} (2x + 2y) dx] dy$
   (d) $\int_0^4 [\int_0^y 3\sqrt{y^2 + 9} dx] dy$.
   Ans. $\int_0^4 [\int_x^4 3\sqrt{y^2 + 9} dx] dy$
   (e) $\int_1^2 [\int_{y^2}^{y^3} dx] dy$. Ans. $\int_1^4 [\int_{\sqrt[3]{x}}^{\sqrt{x}} dy] dx + \int_4^8 [\int_{\sqrt[3]{x}}^2 dy] dx$.
   (f) $\int_0^{\frac{\pi}{2}} [\int_0^{\cos x} y dy] dx$.
   Ans. $\int_0^1 [\int_0^{\cos^{-1} y} y dx] dy$
   (g) $\int_1^{e^3} [\int_0^{\frac{1}{y}} e^{xy} dx] dy$.
   Ans. $\int_0^{\frac{1}{e^3}} [\int_1^{e^3} e^{xy} dy] dx + \int_{\frac{1}{e^3}}^1 [\int_1^{\frac{1}{x}} e^{xy} dy] dx$
   (h) $\int_1^3 [\int_0^{\ln y} y e^x dx] dy$.
   Ans. $\int_0^{\ln 3} [\int_{e^x}^3 y e^x dy] dx$
   (i) $\int_0^1 [\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y dx] dy$.
   Ans. $\int_{-1}^1 [\int_0^{\sqrt{1-x^2}} y dy] dx$.
   (j) $\int_0^\pi [\int_0^x x \cos y dy] dx$.
   Ans. $\int_0^\pi [\int_y^\pi x \cos y dx] dy$
2. Use double integrals to find the area of the region bounded by the given curves and lines.
   (a) The parabola $x = y^2$ and the line $y = x - 2$.
   Ans. $\int_{-1}^2 [\int_{y^2}^{y+2} dx] dy$
   (b) The parabola $y = x - x^2$ and the line $x + y = 0$.
   Ans. $\int_0^2 [\int_{-x}^{x-x^2} dy] dx$
   (c) The axes and the line $2x + y = 2a$ $(a > 0)$.
   Ans. $\int_0^a [\int_0^{2a-2x} dy] dx$
   (d) The y-axis, the line $y = 3x$, and the line $y = 6$.
   Ans. $\int_0^2 [\int_{3x}^6 dy] dx$
   (e) The x-axis, the curve $y = e^{-x}$, and the lines $x = 0$, $x = a$ $(a > 0)$.
   Ans. $\int_0^a [\int_0^{e^{-x}} dy] dx$
   (f) The parabolas $y = x^2$ and $y = 2x - x^2$.
   Ans. $\int_0^1 [\int_{x^2}^{2x-x^2} dy] dx$

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1. Evaluate $\int\int\int_B dxdydz$, where B is the region bounded by the coordinate planes and the plane $x + y + z = 1$.
2. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
3. Evaluate
   (a) $\int_0^1 \int_0^{x^2} \int_0^{xy^3} 18x^3y^2 z dzdydx$
   (b) $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$
   (c) $\int_0^2 \int_0^{\pi} \int_0^{\ln 4} x^3 \cos \frac{y}{2} e^z dzdydx$
   (d) $\int_0^1 \int_0^{\sqrt{3}z} \int_0^{\sqrt{3(y^2+z^2)}} xyz\sqrt{x^2 + y^2 + z^2}dxdydz$
   (e) $\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \sin y^2 dzdydx$
4. Evaluate $\int\int\int_W zdxdydz$, where W is the region bounded by the four planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$, with $x \geq 0$, $y \geq 0$.
5. Evaluate $\int\int\int_W ze^{x+y}dxdydz$, where $W = [0, 1] \times [0, 1] \times [0, 1]$.
6. Evaluate $\int\int\int_W (x^2 + y^2 + z^2)dxdydz$, where W is the region bounded by $x + y + z = a$ $(a > 0)$, $x = 0$, $y = 0$, and $z = 0$.
7. Set up a triple integral for the volume of each of the following solid regions.
   (a) The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes given by $x + y = 1$ and $x + y = 3$
   (b) The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$.
   (c) The region bounded below by the paraboloid $z = x^2+y^2$ and above by the sphere $x^2+y^2+z^2 = 6$.
8. Set up triple integrals that compute the volumes of the given regions.
   (a) The region in the first octant bounded by the cylinder $x = 4-y^2$ and the planes $y = z$, $x = 0$, $z = 0$.
   (b) The region above the xy-plane bounded by the surfaces $z^2 = 16y$, $z^2 = y$, $y = x$, $y = 4$, and $x = 0$.
   (c) The region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$.
   (d) The region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $(a > 0, b > 0, c > 0)$.
   (e) The region bounded by the cylinder $z = 4 - y^2$ and the paraboloid $z = x^2 + 3y^2$.
   (f) The region bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ $(a > 0, b > 0, c > 0)$
   (g) The region bounded by the cylinder $x^2+y^2 = 4x$, the xy-plane and the paraboloid $4z = x^2+y^2$.

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1. Use a double integrals in polar coordinates to find the areas of the indicated regions.
   (a) The cardioid $r = a(1+\cos\theta)$ $(a > 0)$.
   Ans. $2\int_0^{\pi}[\int_0^{a(1+\cos\theta)} rdr]d\theta = \frac{3\pi a^2}{2}$
   (b) The circle $r = a$.
   Ans. $\int_0^{2\pi}[\int_0^{a} rdr]d\theta = \pi a^2$
   (c) The circle $r = 2a\sin\theta$.
   Ans. $\int_0^{\pi}[\int_0^{2a\sin\theta} rdr]d\theta = \pi a^2$
   (d) The circle $r = 2a\cos\theta$.
   Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{2a\cos\theta} rdr]d\theta = \pi a^2$
   (e) One loop of $r = a\cos 2\theta$ $(a > 0)$.
   Ans. $2\int_0^{\frac{\pi}{4}}[\int_0^{a\cos 2\theta)} rdr]d\theta = \frac{\pi a^2}{8}$
   (f) One loop of $r = 3\cos 3\theta$.
   Ans. $2\int_0^{\frac{\pi}{6}}[\int_0^{3\cos 3\theta)} rdr]d\theta = \frac{3\pi}{4}$
   (g) The region inside the lemniscate $r^2 = 2a^2\cos 2\theta$ and outside the circle $r = a$ $(a > 0)$.
   Ans. $4\int_0^{\frac{\pi}{6}}[\int_a^{a\sqrt{2\cos 2\theta}} rdr]d\theta = \frac{a^2}{3}(3\sqrt{3}-\pi)$
   (h) The region inside $r = \tan\theta$ and between $\theta = 0$ and $\theta = \frac{\pi}{4}$.
   Ans. $\int_0^{\frac{\pi}{4}}[\int_0^{\tan\theta} rdr]d\theta = \frac{1}{8}(4-\pi)$
   (i) The region inside the cardioid $r = 2a(1+\cos\theta)$ and outside the circle $r = 3a$ $(a > 0)$.
   Ans. $2\int_0^{\frac{\pi}{3}}[\int_{3a}^{2a(1+\cos\theta)} rdr]d\theta = \frac{a^2}{2}(9\sqrt{3}-2\pi)$
   (j) The region inside the cardioid $r = 1+\cos\theta$ and to the right of the line $x = \frac{3}{4}$.
   Ans. $2\int_0^{\frac{\pi}{3}}[\int_{\frac{3}{4}\sec\theta}^{1+\cos\theta} rdr]d\theta = \frac{1}{16}(8\pi+9\sqrt{3})$

2. Write the integral in the form $\int_\alpha^\beta\int_{r_1(\theta)}^{r_2(\theta)} zrdrd\theta$.
   (a) $\int_0^2\int_0^{\sqrt{4-x^2}} zdydx$.
   Ans. $\int_0^{\frac{\pi}{2}}[\int_0^2 zrdr]d\theta$
   (b) $\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} zdydx$.
   Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^3 zrdr]d\theta$
   (c) $\int_{-1}^0\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} zdxdy$.
   Ans. $\int_\pi^{2\pi}[\int_0^1 zrdr]d\theta$
   (d) $\int_0^1\int_{x^2}^{x} zdydx$.
   Ans. $\int_0^{\frac{\pi}{4}}[\int_0^{\sec\theta\tan\theta} zrdr]d\theta$
   (e) $\int_0^4\int_0^{\sqrt{4-(x-2)^2}} zdydx$.
   Ans. $\int_0^{\frac{\pi}{2}}[\int_0^{4\cos\theta} zrdr]d\theta$
   (f) $\int_0^2\int_0^{\sqrt{2y-y^2}} zdxdy$.
   Ans. $\int_0^{\frac{\pi}{2}}[\int_0^{2\sin\theta} zrdr]d\theta$

3. Use a triple integral in cylindrical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
Ans. $2\int_0^{2\pi}[\int_0^a(\int_0^{\sqrt{a^2-r^2}} dz)rdr]d\theta = \frac{4}{3}\pi a^3$

4. Use cylindrical coordinates to solve the following problems.
   (a) Find the volume of the solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane. Ans. $\int_0^{2\pi}[\int_0^1(\int_0^{1-r^2} dz)rdr]d\theta = \frac{\pi}{2}$
   (b) A cylindrical hole of radius a is bored through the center of a solid sphere of radius $2a$. Find the volume of the hole. Ans. $2\int_0^{2\pi}[\int_0^{2a}(\int_0^{\sqrt{4a^2-r^2}} dz)rdr]d\theta = 4\sqrt{3}\pi a^3$
   (c) Find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$. Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{2\cos\theta}(\int_{r^2}^{2r\cos\theta} dz)rdr]d\theta = \frac{\pi}{2}$
   (d) Find the volume of the region bounded above by the plane $z = x$ and below by the paraboloid $z = x^2 + y^2$.
   Ans. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\int_0^{\cos\theta}(\int_{r^2}^{r\cos\theta} dz)rdr]d\theta = \frac{\pi}{32}$
   (e) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$ $(a > 0)$.
   Ans. $\int_0^{\pi}[\int_0^a(\int_{\frac{r^2}{a}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7)$
   (f) Find the volume of the region inside the cylinder $r = a\sin\theta$ which is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ where $b < a$.
   Ans. $\int_0^{\pi}[\int_0^{a\sin\theta}(\int_{b\sqrt{1-\frac{r^2}{a^2}}}^{2a^2-r^2} dz)rdr]d\theta = \frac{1}{6}\pi a^3(8\sqrt{2} - 7) = \frac{1}{9}a^2(a - b)(3\pi - 4)$

5. Determine the new region that we get by applying the given transformation to the region R.
R is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.
Note that the given region R is a triangle with vertices $(\frac{3}{2}, \frac{5}{2})$, $(-\frac{7}{2}, -\frac{5}{2})$, and $(4, 0)$. The new region is the triangle with vertices $(-6, -1)$, $(4, -1)$, and $(4, 4)$.

6. Evaluate $\int\int_R dA$, where R is the trapezoidal region with vertices given by $(0, 0), (5, 0), (\frac{5}{2}, \frac{5}{2})$, and $(\frac{5}{2}, -\frac{5}{2})$, by making the change of variables $x = 2u + 3v$ and $y = 2u - 3v$.
Note $u = \frac{x+y}{4}$ and $u = \frac{x-y}{6}$. The region in the uv-plane is a square with vertices $(0, 0)$, $(\frac{5}{4}, 0)$, $(\frac{5}{4}, \frac{5}{6})$, and $(0, \frac{5}{6})$. Ans. $\frac{5}{4} \times \frac{5}{6} = \frac{25}{24}$.

7. Let R be the region bounded by the line $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.
Evaluate $\int\int_R 3xydxdy$ by making the change of variables
$x = \frac{1}{3}(2u + v)$, $y = \frac{1}{3}(u - v)$.
We note that $J = \frac{1}{3}$ and $u = x + y$ and $v = x - 2y$. From the given equations, it follows that $1 \le u(= x + y) \le 4$ and $-4 \le v\,(= x - 2y) \le 0$. Thus $\int\int_R 3xydxdy = \int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$.
Alternatively, the given region is a rectangle with vertices $(-\frac{2}{3}, \frac{5}{3})$,$(\frac{2}{3}, \frac{1}{3})$, $(\frac{8}{3}, \frac{4}{3})$, and $(\frac{4}{3}, \frac{8}{3})$ and the corresponding region in the uv-plane is the rectangle with vertices $(1, 0), (4, 0), (1, -4)$, and $(4, -4) = [1, 4] \times [-4, 0]$. Ans. $\int_1^4[\int_{-4}^0 3(\frac{1}{3}(2u + v))(\frac{1}{3}(u - v))(\frac{1}{3})dv]du = 4$

8. Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral $\int\int_R(x+y)^2\sin^2(x-y)dxdy$ by making the change of variables $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. The corresponding region in the uv-plane is the square with vertices $(1, 1), (3, 1), (1, -1)$, and $(3, -1)$. $J = \frac{1}{2}$
$\int\int_R(x + y)^2\sin^2(x - y)dxdy = \int_{-1}^1[\int_1^3 u^2\sin^2 v\frac{1}{2}du]dv = \frac{13}{6}(2 - \sin 2)$

9. Let R be the region bounded by the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(3, 3)$, and $(7, 3)$. Evaluate the integral $\int\int_R y(x - y)dxdy$ by making the change of variables $x = u + v$, $y = u$.
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10. Use a triple integral in spherical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
Ans. $\int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi d\theta = \frac{4}{3}\pi a^3$.

11. Find the volume of the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.
Ans. $\int_0^{2\pi} \int_0^{\alpha} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi d\theta = \frac{4}{3}\pi a^3(1 - \cos\alpha)$.

12. If $0 < b < a$ and $0 < \alpha < \pi$, find the volume of the region bounded by the concentric spheres $\rho = b$, $\rho = a$ and the cone $\phi = \alpha$.
Ans. $\int_0^{2\pi} \int_0^{\alpha} \int_b^a \rho^2 \sin\phi \, d\rho \, d\phi d\theta = \frac{2\pi}{3}(a^3 - b^3)(1 - \cos\alpha)$.

13. Find the volume of the solid region bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.
Ans. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi d\theta = 9\pi(2 - \sqrt{2})$.

14. Evaluate $\int\int\int_E 16z dx dy$, where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.
Ans. $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 (16\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi d\theta = 4\pi$.

15. Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) dz dx dy$ into an integral in spherical coordinates. Ans.
$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}} (\rho^2)\rho^2 \sin\phi \, d\rho \, d\phi d\theta$.

16. Convert $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_a^{a+\sqrt{a^2-x^2-y^2}} x dz dy dx$
into an integral in spherical coordinates.
Ans. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{a \sec phi}^{2a\cos\phi} \rho^3 \sin^2\phi \cos\theta d\rho \, d\phi d\theta$.

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1. Evaluate $\int_C (x-3y)ds$, where C is the line segment from $(0,0)$ to $(1,2)$ and C is parameterized as:
   (a) $x=t$, $y=2t$, $0 \le t \le 1$.
   (b) $x=\sin t$, $y=2\sin t$, $0 \le t \le \frac{\pi}{2}$.
2. Evaluate $\int_C (x^2-y+3z)ds$, where C is the line segment from $(0,0,0)$ to $(1,2,1)$. [Answer. $\int_0^1 (t^2-2t+3t)\sqrt{6}dt = \frac{5\sqrt{6}}{6}$].
3. Evaluate $\int_C xds$, where C is the line segment from $(0,0)$ to $(1,1)$ and $y=x^2$ from $(1,1)$ to $(0,0)$. [Answer. $\int_0^1 t\sqrt{2}dt + \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{\frac{3}{2}}-1)$].
4. Evaluate the line integrals $\int_C (x-3y)dx$, and $\int_C (x-3y)dy$, if C is the part of the parabola $x=y^2$ that joins the points $(1,1)$ and $(4,2)$ [Answer. $\int_1^2 (t^2-3t)2tdt = \frac{-13}{2}$ and $\int_1^2 (t^2-3t)dt = \frac{-13}{6}$].
5. Evaluate the line integral $\int_C (y^2dx - x^2dy)$ along the two curves given below:
   (a) C_1: The parabola $x=t$, $y=t^2$ joining the two points $(0,0)$ and $(2,4)$. [Answer. $\int_0^2 (t^4 - 2t^3)dt = \frac{-8}{5}$].
   (b) C_2: The line $x=t$, $y=2t$ joining the two points $(0,0)$ and $(2,4)$. [Answer. $\int_0^2 2t^2dt = \frac{16}{3}$].
6. Evaluate the line integral $\int_C xy^2dx - (x+y)dy$, where C is
   (a) The straight line segment from $(0,0)$ and $(1,2)$. [Answer. $\int_0^1 (4t^3-6t)dt = -2$].
   (b) The parabolic path from $(0,0)$ and $(2,4)$. [Answer. $\int_0^1 (4t^5-4t^2-8t^3)dt = \frac{-8}{3}$].
   (c) The broken line from $(0,0)$ to $(1,0)$ to $(1,2)$. [Answer. $-\int_0^2 (1+t)dt = -4$].
7. Evaluate the line integral $\int_C xy^2dx - (x+y)dy$, where C is the broken line joining the points $(0,0)$, $(1,1)$, $(2,1)$ in this order.
8. Evaluate the line integral $\int_C \frac{dx}{y} + \frac{dy}{x}$, where C is the part of the hyperbola $xy=4$ from $(1,4)$ to $(4,1)$. [Answer. $\int_1^4 (\frac{t}{4} - \frac{4}{t^3})dt = 0$].
9. Evaluate the line integral $\int_C xdx + x^2dy$ from $(-1,0)$ to $(1,0)$.
   (a) Along the x-axis. [Answer. $\int_{-1}^1 tdt = 0$].
   (b) Along the semicircle $y=\sqrt{1-x^2}$ [Answer. $\int_0^\pi (-\sin t + 1 - \sin^t)dt = 0$].
   (c) Along the broken line from $(-1,0)$ to $(0,1)$ to $(1,1)$. [Answer. $\int_{-1}^0 (t+t^2)dt + \int_0^1 tdt - \int_0^1 dt = -\frac{1}{6} + \frac{1}{2} - 1 = -\frac{2}{3}$].
10. Evaluate the line integral $\int_C ydx + (x+2y)dy$ from $(1,0)$ to $(0,1)$, where C is
    (a) the arc of the circle $x=\cos t$, $y=\sin t$; [Answer. $\int_0^{\frac{\pi}{2}} (\cos^t - \sin^2 t + 2\sin t\cos t)dt = 1$].
    (b) the straight line segment $y=1-x$; [Answer. $\int_1^0 (-1)dx = 1$].
    (c) the broken line from $(1,0)$ to $(1,1)$ to $(0,1)$. [Answer. $\int_0^1 (1+2y)dy + \int_1^0 dx = 1$].
11. Evaluate $\int_C (3x+4y)dx + (2x+3y^2)dy$, where C is the circle $x^2+y^2=4$ traversed counterclockwise from $(2,0)$. [Answer. $x=2\cos t, y=2\sin t$, $0 \le t \le 2\pi$, $\int_0^{2\pi} ((24\sin^2 t - 12\sin t)\cos t + (8 - 24\sin^2 t))dt = -8\pi$].
12. Evaluate $\int_C (x+2)ds$, where C is the curve represented by $r^{\rightarrow} = ti^{\rightarrow} + \frac{4}{3}t^{\frac{3}{2}}j^{\rightarrow} + \frac{1}{2}t^2k^{\rightarrow}$, $0 \le t \le 6\pi$.

13. Evaluate $\int_C F^{\rightarrow} \cdot ds^{\rightarrow}$, if $F^{\rightarrow}(x, y, z) = xy^2 i^{\rightarrow} + x^2 z j^{\rightarrow} - (y - x)k^{\rightarrow}$, C is the curve $c^{\rightarrow}(t) = ti^{\rightarrow} + t^2 j^{\rightarrow} + t^3 k^{\rightarrow}$, $0 \leq t \leq 1$.

14. Compute $\int_C F^{\rightarrow} \cdot ds^{\rightarrow}$ if $F^{\rightarrow}(x, y) = (x + y)i^{\rightarrow} + (y^2 - x)j^{\rightarrow}$, where C is the closed curve that begins at $(1, 0)$, proceeds along the upper half of the unit circle to $(-1, 0)$, and returns to $(1, 0)$ along the x-axis.

15. (a) Evaluate $\int_S xy^4 ds$, where C is the right half of the circle, $x^2+y^2 = 1$. [Answer. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos^t)(4\sin t)^4(4)dt = \frac{8192}{5}$].
    (b) Evaluate $\int_S 4x^3 ds$, were C is the line segment from (-2,-1) to (1,2). [Answer. $\int_{-2}^{0} 4(-2 + 3t)^3\sqrt{18}dt = -15\sqrt{2}$].
    (c) Evaluate $\int_S 4x^3 ds$, were C is the line segment from (1,2) to (-2,-1). [Answer. $\int_0^1 4(1 - 3t)^3\sqrt{18}dt = -15\sqrt{2}$].

16. Evaluate $\int_C xds$ for each of the following curves. (a) C_1: $y = x^2$, $-1 \leq x \leq 1$ (b) C_2: The line segment from(-1,1) to (1,1). (c) C_3: The line segment from(1,1) to (-1,1). [Answer: 0 for each part (a)-(c)].

17. Evaluate $\int_C xyzds$, where C is the helix given by, $x = \cos t$, $y = \sin t$, $z = 3t$, $0 \leq t \leq 4\pi$.

18. Evaluate $\int_C \sin(\pi y)dy + yx^2 dx$, where C is the line segment from (0,2) to (1,4). [Answer. $\frac{7}{6}$].

19. Evaluate $\int_C \sin(\pi y)dy + yx^2 dx$, where C is the line segment from (1,4) to (0,2). [Answer. $-\frac{7}{6}$].

20. Evaluate $\int_C ydx + xdy + zdx$, where C is given by, $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \leq t \leq 2\pi$. [Answer. $\int_0^{2\pi} (-\sin^2 t + \cos^2 + 2t^3)dt = 8\pi^4$].

21. Evaluate $\int_C (2xydx + x^2 dy)$ if:
    (a) C consists of the line segments from $(3, 1)$ to $(5, 1)$ and from $(5, 1)$ to $(5, 6)$.
    (b) C is the line segments from $(3, 1)$ to $(5, 6)$.
    (c) C is the part of the parabola $x = 2t + 1$, $y = 2t^2 - t$, $1 \leq t \leq 2$. [Answer: 141, for (a), (b), (c)]

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EXERCISES Page 528 1, 2, 3, 5, 13  EXERCISES Page 558 2, 3, 4, 7, 9,
13, 14
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1. Evaluate $\int_C (2xydx + x^2dy)$ if:
   (a) C consists of the line segments from $(3,1)$ to $(5,1)$ and from $(5,1)$ to $(5,6)$.
   (b) C is the line segments from $(3,1)$ to $(5,6)$.
   (c) C is the part of the parabola $x = 2t+1$, $y = 2t^2 - t$, $1 \le t \le 2$.
   Answer: 141 (Note: $(2xy, x^2) = \nabla(x^2y)$).

Definition; A vector field F is called conservative if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the potential function for F.

2. Find a potential function for
   (a) $\mathbf{F}(x,y) = (2xy + 24x)\mathbf{i} + (x^2 + 16y)\mathbf{j}$. (Answer: $x^2y + 12x^3 + 16y + C$)
   (b) $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$. (Answer: $x^2y - \frac{y^2}{2} + C$)
   (c) $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$. (Answer: $x^2y + z^2y + C$)
   (d) $\mathbf{F}(x,y,z) = (y\cos x + 2xe^y)\mathbf{i} + (\sin x + x^2e^y + 4)\mathbf{j}$. (Answer: $y\sin x x^2e^y + 4y + C$)
   (e) $\mathbf{F}(x,y) = (x^2 - yx)\mathbf{i} + (y^2 + xy)\mathbf{j}$.
   (f) $\mathbf{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + 2y)\mathbf{j}$. (Answer: $x^2e^{xy} + y^2 + C$)
   (g) $\mathbf{F}(x,y,z) = 2xy^3z^4\mathbf{i} + (3x^2y^2z^4)\mathbf{j} + 4x^2y^3z^3\mathbf{k}$. (Answer: $x^2y^3z^4$)
   (h) $\mathbf{F}(x,y,z) = 2x\cos y - 2z^3\mathbf{i} + (3 + 2ye^z - x^2\sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$. (Answer: $y^2e^z - 2xz^3 + x^2\cos y + 3y + C$)

The Fundamental Theorem of Line Integrals:
Let **F** be a conservative vector field with potential function f, and C be any smooth curve starting at the point A and ending at the point B. Then $\int_C \mathbf{F}.d\mathbf{r} = f(B) - f(A)$.

3. Evaluate $\int_C \mathbf{F}.d\mathbf{r}$, where C is a piecewise smooth curve from $(-1,4)$ to $(1,2)$ and $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$. (Answer: $x^2y - \frac{y^2}{2} + C$, 4)

4. Evaluate $\int_C \mathbf{F}.d\mathbf{r}$, where C is a piecewise smooth curve from $(1,1,0)$ to $(0,2,3)$ and $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$. (Answer: $x^2y + z^2y + C$, 17)

5. Use Green's theorem to evaluate the line integral $\oint[xydx + x^2y^3dy]$, where C is the triangle with vertices $(0,0), (1,0)$, and $(1,2)$.
   (Answer: $P = xy$, $Q = x^2y^3$, the region enclosed is $0 \le x \le 1, 0 \le y \le 2x$, $\int_0^1 \int_0^{2x} (Q_x - P_y)dydx = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \frac{2}{3}$).

6. Use Green's theorem to evaluate the line integral $\oint[y^3dx + x^3dy]$, where C is the circle of radius 2 centered at $(0,0)$.
   (Answer: $P = y^3$, $Q = x^3$, the region enclosed is $0 \le \theta \le 2\pi, 0 \le r \le 2$, $\int_0^{2\pi} \int_0^2 (Q_x - P_y)rdrd\theta = \int_0^{2\pi} \int_0^2 (3r^2\cos^2\theta - 3r^2\sin^2\theta)rdrd\theta = 0$).

7. Use Green's theorem to evaluate the line integral $\oint[(-2xy+y^2)dx+x^2dy]$, where C is the boundary of the region R enclosed by $y = 4x$ and $y = 2x^2$.
   (Answer: $P = -2yx + y^2$, $Q = x^2$, the region enclosed is $0 \le x \le 2, 2x^2 \le y \le 4x$, $\int_0^2 \int_{2x^2}^{4x} (Q_x - P_y)dydx = \int_0^2 \int_{2x^2}^{4x} (4x - 2y)dydx = \frac{-32}{5}$).

8. Evaluate the line integral $\oint(3x-y)dx+(x+5y)dy$ around the unit circle $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$.
(Answer: $P=3x-y$, $Q=x+5y$, the region enclosed is $0\le\theta\le 2\pi, 0\le r\le 2$, $\int_0^{2\pi}\int_0^1(Q_x-P_y)rdrd\theta=\int_0^{2\pi}\int_0^1(1-(-1))rdrd\theta=2\pi$).

9. Evaluate the line integral $\oint[(e^{-x^2}+y^2)dx+(\ln y-x^2))dy]$, where C is the square with vertices $(0,0),(1,0),(1,1),(0,1)$.
(Answer: $P=e^{-x^2}+y^2$, $Q=\ln y-x^2$, the region enclosed is $0\le x\le 1, 0\le y\le 1$, $\int_0^1\int_0^1(Q_x-P_y)dydx=\int_0^1\int_0^1(-2x-2y)dydx=-2$).

10. Evaluate the line integral $\oint(2y+\sqrt{1+x^5})dx+(5x-e^{y^2})dy$ around the circle $x^2+y^2=4$.
(Answer: $P=2y+\sqrt{1+x^5}$, $Q=5x-e^{y^2}$, the region enclosed is $0\le\theta\le 2\pi, 0\le r\le 2$, $\int_0^{2\pi}\int_0^2(Q_x-P_y)rdrd\theta=\int_0^{2\pi}\int_0^2(5-2)rdrd\theta=12\pi$).

11. If R is any region to which Green's Theorem is applicable, show that the area of R is given by the formula
$A=\frac{1}{2}\oint_C -ydx+xdy$, or
$A=\oint_C xdy$, or
$A=\oint_C(-y)dx$.

12. Use Green's theorem to find the area of the region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
(Answer. Parametric equations for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: $x=a\cos t$, $y=b\sin t$, $0\le t\le 2\pi$. Area $=\frac{1}{2}\oint_C -ydx+xdy=\frac{1}{2}\int_0^{2\pi}[(-b\sin t)(-a\sin tdt)+(a\cos t)(b\cos tdt)]=\pi ab$.)

13. Use Green's theorem to find the area of the region enclosed by the circle $x^2+y^2=a^2$.
(Answer. Parametric equations for the circle $x^2+y^2=a^2$: $x=a\cos t$, $y=a\sin t$, $0\le t\le 2\pi$. Area $=\frac{1}{2}\oint_C -ydx+xdy=\frac{1}{2}\int_0^{2\pi}[(-a\sin t)(-a\sin tdt)+(a\cos t)(a\cos tdt)]=\pi a^2$.)

14. Use Green's theorem to find the area of the region enclosed by $y=x^2$ and $y=x+2$.
(Answer. Parametric equations for the boundary of the region: $C_1: x=t,\ y=t^2,\ -1\le t\le 2$; $C_2: x=-3t+2,\ y=-3t+4$ Area $=\frac{1}{2}\oint_C -ydx+xdy=\frac{1}{2}\int_{-1}^2[(-t^2)(dt)+(t)(2tdt)]+\frac{1}{2}\int_0^1[(3t-4)(-3dt)+(-3t+2)(-3dt)]\frac{9}{2}$.)

15. Use Green's theorem to find the area of the region enclosed by $y=x^2-1$ and $y=0$.
(Answer. Parametric equations for the boundary of the region: $C_1: x=t, y=1-t^2, -1\le t\le 1$, $C_2: x=-t,\ y=0,\ -1\le t\le 1$, Area $=\frac{1}{2}\oint_{C_1} -ydx+xdy+\frac{1}{2}\oint_{C_2} -ydx+xdy=\frac{1}{2}\int_{-1}^1[(-(1-t^2)(dt)+(t)(-2tdt)]+\frac{1}{2}\int_{-1}^1[(0)(dt)+(-t)(0dt)]$.)

16. Use Green's theorem to find the area under one arc of the cycloid $x=2\pi t-\sin 2\pi t$ and $y=1-\cos 2\pi t$; $0\le t\le 1$.
(Answer. Parametric equations for the boundary of the region: $C_1: x=t,\ y=0,\ 0\le t\le 2\pi$, $C_2: x=2\pi t-\sin 2\pi t,\ y=1-\cos 2\pi t,\ 0\le t\le 1$, Area $=\frac{1}{2}\oint_{C_1} -ydx+xdy+\frac{1}{2}\oint_{C_2} -ydx+xdy=\frac{1}{2}\int_0^1[(-(0)(dt)+(t)(0dt)]-\frac{1}{2}\int_0^1[(-(2\pi t-\sin 2\pi t)((2\pi-2\pi\cos 2\pi t)dt)+(2\pi t-\sin 2\pi t)(2\pi\sin 2\pi tdt)]=3\pi$.)

EXERCISES Page 471 1, 2, 3, 5, 13, 21, 22 EXERCISES Page 481 2, 3, 4, 5(a)

1. **Let S be a smooth parametric surface $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ defined over the region D in the uv-plane. Let (u_0, v_0) be a point in D. Note $\mathbf{r_u}(x_0, y_0) = \frac{\partial x}{\partial u}(u_0, v_0) + \frac{\partial y}{\partial u}(u_0, v_0) + \frac{\partial z}{\partial u}(u_0, v_0)$. A normal vector at the point $(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), y(u_0, v_0))$ is given by $N = \mathbf{r_u}(x_0, y_0) \times \mathbf{r_v}(x_0, y_0)$.**

2. **Let S be a smooth parametric surface $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ defined over the region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the surface area of S is given by $\int\int_S dS = \int\int_D \| \mathbf{r_u} \times \mathbf{r_v} \| \, dA$. For a surface given by $z = f(x,y)$, $\mathbf{r}(u,v) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$, $\| \mathbf{r_u} \times \mathbf{r_v} \| = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}$**

3. Find the surface area of the unit sphere given by $\mathbf{r}(u,v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k}$, where the domain D is given by $0 \le u \le \pi$, $0 \le v \le 2\pi$. ($|\mathbf{r_u} \times \mathbf{r_v}| = |\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}| = \sin u$ (Answer: 4π))

4. Find the surface area of the torus given by $\mathbf{r}(u,v) = (2 + \cos u)\cos v\mathbf{i} + (2 + \cos u)\sin v\mathbf{j} + \sin u\mathbf{k}$, where the domain D is given by $0 \le u \le 2\pi$, $0 \le v \le \pi$. (($|\mathbf{r_u} \times \mathbf{r_v}| = 2 + \cos u$ (Answer; $8\pi^2$))

5. Find the surface area of the portion of sphere, center (0,0), of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.
   $\int_0^{2\pi} \int_0^{\frac{\pi}{3}} 16 \sin\phi \, d\phi d\theta = 16\pi$.

6. Let $\Phi(u,v) = (u - v, u + v, uv)$ and let D be a unit disk in the uv plane. Find the area of $\Phi(D)$.

7. Find the area of the portion of the unit sphere that is cut out by the cone $z \ge \sqrt{x^2 + y^2}$.

8. Find the area of the surface defined by $x + y + z = 1$, $x^2 + 2y^2 \le 1$.

9. Find the area of the graph of the function $f(x,y) = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}})$ that lies over the domain $[0,1] \times [0,1]$.

10. Express the surface area of the following graphs over the indicated region D as a double integral. Do not evaluate.
    (a) $(x + 2y)^2$; $D = [-1, 2] \times [0, 2]$
    (b) $xy + \frac{x}{y+1}$; $D = [1, 4] \times [1, 2]$
    (c) $xy^3 e^{x^2 y^2}$; D = unit circle centered at the origin
    (d) $y^3 \cos^2 x$; D = triangle with vertices $(-1, 1), (0, 2)$, and $(1, 1)$.

11. **Let S be a surface with equation $z = g(x,y)$ and let R be its projection onto the xy-plane. If g, g_x, g_y are continuous on R and f is continuous on S, then the surface integral of f over S is given by**
    $\int\int_S f(x,y,z)dS = \int\int_R f(x,y,g(x,y))\sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}dA$.
    If $y = g(x,z)$ and let R be its projection onto the xz-plane, then $\int\int_S f(x,y,z)dS = \int\int_R f(x, g(x,z), z)\sqrt{1 + [g_x(x,z)]^2 + [g_z(x,z)]^2}dA$.
    If $x = g(y,z)$ and let R be its projection onto the yz-plane, then $\int\int_S f(x,y,z)dS = \int\int_R f(g(y,z), y, z)\sqrt{1 + [g_y(y,z)]^2 + [g_z(y,z)]^2}dA$.

12. Evaluate the surface integral $\int\int_S (y^2 + 2yz)dS$, where S is the first-octant portion of the plane $2x + y + 2z = 6$.
    $3\int_0^3 \int_0^{2(3-x)} y(3 - x)dydx = \frac{243}{2}$.

13. Evaluate $\int\int_S 6xydS$,where S is the first-octant portion of the plane $x+y+z=1$. $6\sqrt{3}\int_0^1\int_0^{1-y}(y-y^2-zy)dzdy = \frac{\sqrt{3}}{4}$.

14. Evaluate $\int\int_S(x^2+y^2)dS$, where S is the part of the surface of the paraboloid $z=f(x,y)=1-x^2-y^2$ that lies above the xy-plane.
$\int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}rdrd\theta$.

15. Evaluate $\int\int_S z^2dS$, where S is the portion of the cone $z=\sqrt{x^2+y^2}$ for which $1\le x^2+y^2\le 4$.
$\sqrt{2}\int_0^{2\pi}\int_1^2 r^3drd\theta$.

16. Evaluate $\int\int_S(x+y+z)dS$, where S is the portion of the plane $x+y=1$ in the first octant for which $0\le z\le 1$.
$\sqrt{2}\int_0^1\int_0^1(1+z)dxdz = \frac{3\sqrt{2}}{2}$.

The second method of evaluating a surface integral is for those surfaces that are given by the parameterization, $\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$.
$\int\int_S f(x,y,z)dS = \int\int_R f(\mathbf{r}(u,v)) \parallel \mathbf{r_u}\times\mathbf{r_v} \parallel dA$, **where D is the range of the parameters that trace out the surface S.**

17. Evaluate the surface integral $\int\int_S(x+z)dS$, where S is the first-octant portion of the cylinder $y^2+z^2=9$ between $x=0$ and $x=4$.
In parametric form the surface is given by $\mathbf{r}(x,\theta)=x\mathbf{i}+3\cos\theta\mathbf{j}+3\sin\theta\mathbf{k}$, $0\le x\le 4$, $0\le\theta\le\frac{\pi}{2}$.
$\parallel \mathbf{r_x}\times\mathbf{r}_\theta \parallel = 3$. $\int_0^4\int_0^{\frac{\pi}{2}}(3x+9\sin\theta)d\theta dx = 12\pi+4$.

18. Evaluate the surface integral $\int\int_S zdS$, where S is the upper half of a sphere of radius 2.
$\mathbf{r}(\theta,\phi)=2\sin\phi\cos\theta\mathbf{i}+2\sin\phi\sin\theta\mathbf{j}+2\cos\phi\mathbf{k}$. $\parallel \mathbf{r}_\theta\times\mathbf{r}_\phi \parallel = 4\sin\phi$. $\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(4\sin 2\phi)d\phi d\theta = 8\pi$.

19. Evaluate $\int\int_S ydS$, where S is the portion of the cylinder $x^2+y^2=3$ that lies between $z=0$ and $z=6$. $\mathbf{r}(z,\theta)=\sqrt{3}\cos\theta\mathbf{i}+\sqrt{3}\sin\theta\mathbf{j}+z\mathbf{k}$, $0\le z\le 6$, $0\le\theta\le 2\pi$. $\parallel \mathbf{r_z}\times\mathbf{r}_\theta \parallel = \sqrt{3}$.
$\int_0^{2\pi}\int_0^6 \sin\theta dzd\theta = 0$.

20. Evaluate $\int\int_S xyzdS$, where S is the triangle with vertices $(1,0,0),(0,2,0)$, and $(0,1,1)$.

21. Evaluate $\int\int_S zdS$, where S is the upper hemisphere of radius a; that is, the set of (x,y,z) with $z=\sqrt{a^2-x^2-y^2}$.

22. Evaluate $\int\int_S(x+y+z)dS$, where S is the boundary of the unit ball B; that is, S is the set of (x,y,z) with $x^2+y^2+z^2=1$.

23. Compute the area of the portion of the cone $x^2+y^2=z^2$, with $z^2\ge 0$ that is inside the sphere $x^2+y^2+z^2=2Rz$, where R is a positive constant.
$\parallel r_\rho\times r_\theta \parallel = \sqrt{2}\rho$

1. **Stokes's Theorem** Let S be an oriented surface with unit normal vector $\mathbf{n}$, bounded by a piecewise smooth simple closed curve C. If $\mathbf{F}$ is a vector field whose component functions have continuous partial derivatives on an open region containing S and C, then
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S(\bigtriangledown\times\mathbf{F})\cdot\mathbf{n}\,dS$.

2. Let C be the the oriented triangle from $(3,0,0)$ to $(0,3,0)$ to $(0,0,6)$ to $(3,0,0)$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=-y^2\mathbf{i}+z\mathbf{i}+x\mathbf{k}$.
(The upward normal to $z=g(x,y)$ is given by $-g_x(x,y)\mathbf{i}-g_y(x,y)\mathbf{j}+\mathbf{k}$ and the downward normal by $g_x(x,y)\mathbf{i}+g_y(x,y)\mathbf{j}-\mathbf{k}$)
$\bigtriangledown\times\mathbf{F}=(-1,-1,2y)$. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S(\bigtriangledown\times\mathbf{F})\cdot\mathbf{n}dS = \int_0^3\int_0^{3-y}(2y-4)dxdy = -9$.
$C_1 : \mathbf{r}_1(t)=(3-t,t)$; $0\le t\le 3$,
$C_2 : \mathbf{r}_2(t)=(6-t,2t-6)$; $3\le t\le 6$,
$C_3 : \mathbf{r}_3(t)=(t-6,18-2t)$; $6\le t\le 9$.
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}(\mathbf{r_1(t)})\cdot\mathbf{r}_1'dt + \int_{C_2}\mathbf{F}(\mathbf{r_2(t)})\cdot\mathbf{r}_2'dt + \int_{C_3}\mathbf{F}(\mathbf{r_3(t)})\cdot\mathbf{r}_3'dt = 9-9-9=-9$.

3. Verify Stokes's Theorem for $\mathbf{F}(x,y,z) = 2z\mathbf{i} + x\mathbf{i} + y^2\mathbf{k}$, where S is the surface of the paraboloid $z = 4 - x^2 - y^2$ and C is the trace of S in the xy-plane. $\bigtriangledown \times \mathbf{F} = (2y, 2, 1)$. $\int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4x + 4y + 1) dx dy = 4\pi$. $C : \mathbf{r}(t) = (2\cos t, 2\sin t, 0)$, $0 \le t \le 2\pi$. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (0 + 2\cos t(2\cos t) + 0) dt = 4\pi$

4. Let C be the the oriented triangle from $(2,0,0)$ to $(0,2,1)$ to $(0,0,0)$ to $(2,0,0)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -3y^2\mathbf{i} + 4z\mathbf{i} + 6x\mathbf{k}$.
$\bigtriangledown \times \mathbf{F} = (-4, -6, 6y)$. Note that the projection of S onto the xy-plane is is the first quadrant region bounded by the coordinate axes and the line $x + y = 2$. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS = \int_0^2 \int_0^{2-x} (-(-4)0 - (-6)\frac{1}{2} + 6y) dy dx = 14$.

5. Let C be the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$ and give C its counterclockwise orientation as viewed from the positive z-axis. Evaluate $\int_C xy dx + x^2 dy + z^2 dz$. Let $\mathbf{F}(x,y,z) = xy\mathbf{i} + x^2\mathbf{i} + z^2\mathbf{k}$ and let S be the portion of the plane $z = y$ that lies inside the paraboloid $z = x^2 + y^2$. We need $\int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS$. Now $x^2 + y^2 = y$ gives the circular cylinder $r = \sin\theta$. Thus if D is the region in the xy-plane bounded by the circle $r = \sin\theta$, then S is the graph of $z = y$ on D. When we orient S by the normal directed upward, the induced orientation on C is counterclockwise. $\bigtriangledown \times \mathbf{F} = (1, 0, 0)$. $\int_C xy dx + x^2 dy + z^2 dz = \int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS = \int\int_D (-0(0) - 0(1) + x) dA = \int_0^{\pi} \int_0^{\sin\theta} (r\cos\theta) r dr d\theta = 0$.

6. Verify Stokes's theorem for $\mathbf{F}(x,y,z) = y\mathbf{i} - \mathbf{j}$, where S is the paraboloid $z = x^2 + y^2$ and its boundary C is the circle $x^2 + y^2 = 1, z = 1$. Give C its counterclockwise orientation as viewed from the positive z-axis.
$C : x = \cos t, y = \sin t, z = 1; 0 \le t \le 2\pi$.
$\oint \mathbf{F} \cdot d\mathbf{r} = \oint y dx - x dy = \int_0^{2\pi} \sin t(-\sin t) dt - \cos t \cos t dt = -2\pi$.
$\bigtriangledown \times \mathbf{F} = (0, 0, -2)$ and consistent orientation with C, the outer unit normal $\mathbf{n}$ is given by $\mathbf{n} = \frac{-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$. Thus $\int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS = \int\int_S \frac{-2}{\sqrt{4x^2 + 4y^2 + 1}} dS = -2\pi$.

7. Use Stokes's theorem to evaluate $\int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS$, where $\mathbf{F}(x,y,z) = z^2\mathbf{i} + 3xy\mathbf{j} + x^3y^3\mathbf{k}$, and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume S is oriented upwards.
$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$. $\int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)).\mathbf{r}'(t) = (-2\sin t - 24\sin t\cos^2 t) dt = 0$.

8. Use Stokes Theorem to evaluate $\int\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0), (0,1,0)$, and $(0,0,1)$.
$\bigtriangledown \times \mathbf{F} = (0, 2z - 1, 0)$. $\int\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_0^{-x+1} (1 - 2x - 2y) dy dx = \frac{-1}{6}$.

9. Let $\mathbf{F}(x,y,z) = (yze^x + xyze^x)\mathbf{i} + (xze^x)\mathbf{j} + (xye^x)\mathbf{k}$. Prove that the integral of $\mathbf{F}$ around an oriented simple closed curve is zero.
$\bigtriangledown \times \mathbf{F} = \mathbf{0}$.

10. Find the integral of $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} - \mathbf{k}$ around the triangle with vertices from $(0,0,0)$ to $(0,2,0)$ to $(0,0,3)$ to $(0,0,0)$.
$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\bigtriangledown \times \mathbf{F}) \cdot \mathbf{n} dS$. Note $\bigtriangledown \times \mathbf{F} = \mathbf{0}$.

11. Integrate $\bigtriangledown \times \mathbf{F}$, where $\mathbf{F} = (3y, -xz, -yz^2)$, over the portion of the surface $z = \frac{x^2+y^2}{2}$ defined on the disk $x^2 + y^2 \le 4$.
$\bigtriangledown \times \mathbf{F} = (-z^2 + x, 0, -z - 3)$, and $\mathbf{n} = (-z_x, -z_y, 1)$. $\int\int_S (\bigtriangledown \times \mathbf{F}).\mathbf{n} dS = \int_0^{2\pi} \int_0^1 (\frac{1}{4}r^5\cos\theta - r^2\cos^2\theta - \frac{1}{2}r^2 - 3) r dr d\theta = -20\pi$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (6\sin, -4\cos t, -8\sin t) \cdot (-2\sin t, 2\cos t, 0) dt = -20\pi$.

12. Consider the sphere of radius R centered at $(0,0)$. Then $\mathbf{r}(\theta,\phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$; $0 \le \theta \le 2\pi$, $0 \le \theta \le 2\pi$. Then
$\mathbf{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_\phi}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|}$

$= (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)$
$= -\frac{1}{R}\mathbf{r}(\theta,\phi)$.
Thus $\mathbf{n}$ points in the direction opposite to $\mathbf{r}(\theta,\phi)$; that is, towards the origin (points inside the sphere). In this case the inside of the sphere is the positive side of the surface. However, if we want $\mathbf{n}$ to point outside the sphere, then $\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|}$
$= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

13. Let C be the quarter circle from $(0,0,0)$ to $(0,1,0)$ to $(0,0,1)$ to $(0,0,0)$. Let $\mathbf{F}(x,y,z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

14. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the polygon path from $(1,1,0)$ to $(3,1,4)$ to $(1,1,5)$ to $(-1,1,1)$.

15. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve in which the cone $z^2 = x^2 + y^2$ intersects the plane $z = 1$ (Oriented counterclockwise viewed from the positive z-axis), and $\mathbf{F}(x,y,z) = ((\sin x - \frac{y^3}{3})\mathbf{i} + (\cos y + \frac{x^3}{3})\mathbf{j} + xyz\mathbf{k}$.
$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, 1)$; $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $\bigtriangledown \times \mathbf{F} = (0,0,r)$. Answer: $\frac{\pi}{2}$.

$r = 2a\sin\theta$: $x^2 + (y-a)^2 = a^2$, a circle with center (0,a) and radius a.
$r = 2a\cos\theta$: $(x-a)^2 + y^2 = a^2$, a circle with center (a,0) and radius a.

$r = 2 + 2\cos\theta$ (cardioid), $r = 2 + 2\sin\theta$ (cardioid)
$r = 1 + 2\cos\theta$ (inner loop), $r = 1 + 2\sin\theta$ (inner loop)
$r = 3 + 2\cos\theta$ (does not reach the origin), $r = 3 + 2\sin\theta$ (does not reach the origin)

Graph $r = \sin 3\theta$.
Tangents at the origin: Let $r = 0$. Then $\sin 3\theta = 0 \implies 3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi \implies \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.

	θ	$r = \sin 3\theta$
(1):	$0 \longrightarrow \frac{\pi}{6}$	$0 \longrightarrow 1$
(2):	$\frac{\pi}{6} \longrightarrow \frac{\pi}{3}$	$1 \longrightarrow 0$
(3):	$\frac{\pi}{3} \longrightarrow \frac{\pi}{2}$	$0 \longrightarrow -1$
(4):	$\frac{\pi}{2} \longrightarrow \frac{2\pi}{3}$	$-1 \longrightarrow 0$
(5):	$\frac{2\pi}{3} \longrightarrow \frac{5\pi}{6}$	$0 \longrightarrow 1$
(6):	$\frac{5\pi}{6} \longrightarrow \pi$	$1 \longrightarrow 0$

Graph $r = 2\cos 2\theta$.
Tangents at the origin: Let $r = 0$. Then $2\cos 2\theta = 0 \implies \cos 2\theta = 0 \implies 2\theta = \frac{\pi}{2}, \frac{3\pi}{3}, \frac{5\pi}{2}, \frac{7\pi}{2} \implies \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

	θ	$r = 2\cos 2\theta$
(1):	$0 \longrightarrow \frac{\pi}{4}$	$2 \longrightarrow 0$
(2):	$\frac{\pi}{4} \longrightarrow \frac{\pi}{2}$	$0 \longrightarrow -2$
(3):	$\frac{\pi}{2} \longrightarrow \frac{3\pi}{4}$	$-2 \longrightarrow 0$
(4):	$\frac{3\pi}{4} \longrightarrow \pi$	$0 \longrightarrow 2$
(5):	$\pi \longrightarrow \frac{5\pi}{4}$	$2 \longrightarrow 0$
(6):	$\frac{5\pi}{4} \longrightarrow \frac{3\pi}{2}$	$0 \longrightarrow -2$
(7):	$\frac{3\pi}{2} \longrightarrow \frac{7\pi}{4}$	$-2 \longrightarrow 0$
(8):	$\frac{7\pi}{4} \longrightarrow 2\pi$	$0 \longrightarrow 2$

Graph $r^2 = \cos 2\theta$.
$r = \pm\sqrt{\cos 2\theta}$. Note that $0 \le 2\theta \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le 2\theta \le 2\pi$ (since $\cos 2\theta \ge 0$). Thus $0 \le \theta \le \frac{\pi}{4}$ and $\frac{3\pi}{4} \le \theta \le \pi$ Tangents at the origin: Let $r = 0$. Then $\pm\sqrt{\cos 2\theta} = 0 \implies \cos 2\theta = 0 \implies 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. So $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

	θ	$r = 2\sqrt{\cos 2\theta}$	$r = -2\sqrt{\cos 2\theta}$
(1):	$0 \longrightarrow \frac{\pi}{4}$	$1 \longrightarrow 0$	$-1 \longrightarrow 0$
(2):	Use symmetry		

Graph $r^2 = \sin 2\theta$.
$r = \pm\sqrt{\sin 2\theta}$. Note that $0 \le 2\theta \le \pi$ (since $\sin 2\theta \ge 0$). Thus $0 \le \theta \le \pi$. Tangents at the origin: Let $r = 0$. Then $\pm\sqrt{\sin 2\theta} = 0 \implies \sin 2\theta = 0 \implies 2\theta = 0, \frac{\pi}{2}$. So $\theta = 0, \frac{\pi}{4}$

	θ	$r = 2\sqrt{\sin 2\theta}$	$r = -2\sqrt{\sin 2\theta}$
(1):	$0 \longrightarrow \frac{\pi}{4}$	$0 \longrightarrow 1$	$0 \longrightarrow -1$
(2):	Use symmetry		