

Math 329
Fundamentals of Euclidean Geometry
Chapter 8
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This is a sketch of the class. You will need to take notes and fill in the details.

1. TRANSFORMATIONS

1.1. Euclid's Superposition Approach.

Euclid assumed one could move figures around in space without distortion i.e. without changing size or shape.

Such a motion is called a rigid motion.

A rigid motion keeps the size and shape of a figure, but (possibly) moves it in space, and/or rotates it.

Is Euclid's assumption valid in our world?

1.2. Rigid Motions.

- Rotations
- Shifts
- Reflections

1.3. Euclid's proof of Proposition 1.4, the SAS Postulate. Given: $\triangle ABC$ and $\triangle DEF$, with $AB = DE$, $AC = DF$, and $m\angle A = m\angle D$.

Proof.

If $\triangle ABC$ be applied to $\triangle DEF$ and if the point A be placed on the point D and the straight line \overleftrightarrow{AB} on \overleftrightarrow{DE} , then the point B will also coincide with E , because $AB = DE$. The straight line \overleftrightarrow{AC} will coincide with \overleftrightarrow{DF} since $m\angle BAC = m\angle EDF$. Therefore C will coincide with F . B also coincided with E , so the base \overleftrightarrow{BC} will coincide with the base \overleftrightarrow{EF} , and will equal it.

Thus the whole triangle $\triangle ABC$ will coincide with $\triangle DEF$

And so the triangles are congruent. □

Question 1.1. Is Euclid's proof of the SAS postulate considered a proof today? Why or why not?

Definition 1.2.

- A transformation in Absolute Geometry is a function (or mapping) f that associates with each point P another point P' ; we denote this $f(P) = P'$ such that:
 - f is one-to-one (i.e. $P \neq Q \implies f(P) \neq f(Q)$)
 - f is onto (that is, every point in the plane has a preimage under f ; i.e. for each point P , there exists a point S such that $P = f(S)$.)
- A mapping f is said to preserve collinearity if given three collinear points, their images under f are also three collinear points.
- If a transformation preserves collinearity then it is called a linear transformation.

Lemma 1.3. Any function that is one-to-one and onto has an inverse. That is, if f is a one-to-one and onto function then there exists a function f^{-1} such that $f \circ f^{-1} = f^{-1} \circ f = \text{identity}$. That is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Furthermore, the inverse of a one-to-one and onto function is also one-to-one and onto.

Proof. Exercise. □

Notation 1.4.

Translate point P to its Cartesian coordinates (x, y) . Instead of $f(P) = P'$, write $f(x, y) = (x', y')$.

Equivalently:

$$f(x, y) = (f_1(x, y), f_2(x, y))$$

That is, write each of the coordinates of the new point as a separate function of the old ones.

Exercise 1.5. Suppose a transformation is given in coordinate form by $f(x, y) = (2x + y + 1, x + y - 2)$

- Find the images of the three points $A(3, 1)$, $B(4, 6)$, and $C(5, 11)$ under f .
- Find the (unique) preimage of the point $D(2, 2)$.
- Find the inverse transformation f^{-1} .
- Examine the collinearity properties of the points A, B, C, D . What can you say about them?

Definition 1.6.

A transformation f , defined by $f(x, y) = (f_1(x, y), f_2(x, y))$, is linear iff f_1 and f_2 are of the form $f_1(x, y) = ax + by + c$ and $f_2(x, y) = dx + ey + g$ for some real numbers $a, b, c, d, e, g \in \mathbb{R}$.

Note that requiring that a function f be a transformation, i.e. one-to-one and onto, is *independent* from requiring that it be linear.

Theorem 1. If f is a linear transformation and A, B and C are non-collinear points, then $f(A), f(B)$, and $f(C)$ are also non-collinear.

Proof.

□

Theorem 2. A linear transformation maps parallel lines to parallel lines.

Proof.

□

Definition 1.7.

- Given a transformation of the plane f , A is said to be a fixed point for f if $f(A) = A$.
- A transformation of the plane is called the identity mapping iff every point of the plane is a fixed point of the transformation. This transformation is denoted e .

Exercise 1.8. Consider the function $f(x, y) = (3x + 2y - 1, -x + 4y + 2)$

- Is f linear?
- Segment \overline{AB} has endpoints $A(1, 1)$ and $B(3, 5)$. What is the midpoint of segment \overline{AB} ?
- Segment \overline{CD} has endpoints $C(2, 3)$ and $D(-4, -3)$. What is the midpoint of segment \overline{CD} ?
- What are $f(A)$, $f(B)$ and $f(M)$, where $M = (2, 3)$.
- Is $f(M)$ the midpoint of $A'B'$?

Lemma 1.9. *Linear transformations preserve midpoints.*

Proof.

□

Homework

Section 5.1: 1, 2, 3, 4, 6, 7, 8.