Problem of the Month, May 2007

(This will be the last problem of the month until starting again for the month of October! Thank you to everyone who has participated! And be sure to look at all of the fun problems coming next year!)

Let V be a vector space. Let f be a function that inputs pairs of vectors and outputs real numbers. Suppose also that f satisfies, for all $\vec{u}, \vec{v}, \vec{w} \in V$, and $c \in \mathbb{R}$:

$$\begin{aligned} f(\vec{u}+\vec{v},\vec{w}) &= f(\vec{u},\vec{w}) + f(\vec{v},\vec{w}), \\ f(\vec{u},\vec{v}+\vec{w}) &= f(\vec{u},\vec{v}) + f(\vec{u},\vec{w}), \\ f(c\vec{u},\vec{w}) &= f(\vec{u},c\vec{w}) = cf(\vec{u},\vec{w}), \\ f(\vec{u},\vec{v}) &= f(\vec{v},\vec{u}), \text{ and finally,} \\ f(\vec{v},\vec{v}) &= 0. \end{aligned}$$

(f satisfies the same sorts of properties as an inner product, except for the last condition that $f(\vec{v},\vec{v}) = 0$ for all $\vec{v} \in V$). Any function satisfying the first three properties above (but not necessarily the last two) is called *bilinear*.

Show that for any pair of vectors $\vec{v}$ and $\vec{w}$, that $f(\vec{v},\vec{w}) = 0$.