A brief, but incomplete, exhibition of Corey's incompetance

The Brain

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Hello, human participants in Corey's Math 610 class. I watched with horror yesterday as Corey described a proof to you all about compactness. Although I think everything he said was true, I don't think he said it very well. Also, Corey mentioned another theorem that he said had a lot to do with compactness but didn't really make any sense for a little while after that. I followed Corey to his shame closet and, between sobs, he told me that he had confused a theorem from analysis with the one he wanted to share with you. Being a hyper-intelligent lab mouse, I think he was thinking about uniform continuity, and maps from one compact metric space to another metric space. These should not concern us, and even though Corey had a good idea, it's probably not best to talk about this theorem, it doesn't do what Corey wanted it to do. So, I'm taking a break from taking over the world tonight to write this handout for you about compact spaces and their products. Enjoy, and ROCK ON!

Definition 0.1 *Let X be a topological space. We say X is* compact *if any open cover of X has a finite subcover.*

Definition 0.2 *Let X and Y be topological spaces. The following is called the* product topology *on $X \times Y$: The open sets of $X \times Y$ are given as arbitrary unions of sets of the form $U \times V$, where U is open in X and V is open in Y.*

I would say that the open sets should be generated by arbitrary unions *and* finite intersections of the sets given above, but since

$$A \times B \cap U \times V = (A \cap U) \times (B \cap V),$$

we already know that finite intersections of these sets are again of this type. So we really only need arbitrary unions of sets of this type to generate the entire topology.

Theorem 0.3 *The topological spaces X and Y are compact if and only if $X \times Y$ is compact.*

Proof. The projection maps $\pi_1 : X \times Y \to X$ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are continuous. The continuous image of a compact space is compact. So, if $X \times Y$ is compact, then so is $\pi_1(X \times Y) = X$ and $\pi_2(X \times Y) = Y$.

We now prove the other implication. Suppose X and Y are compact. Suppose we are given an open covering of $X \times Y$. We know that every one of the open sets in this cover is a union of "rectangles" of the type $U \times V$ where U is open in X, and V is open in Y. What we will show is that there exists a finite subcover of open sets of this form. For each of these "rectangles", there exists a larger open set that is a member of the original open cover. Thus, the finite subcover will will exhibit will be the larger open sets from which the smaller open rectangles came.

Let $x \in X$. Then the map $y \to (x, y)$ is a homeomorphism from Y to $\{x\} \times Y$ (it is a continuous open map). Thus the slice $\{x\} \times Y$ is compact as well, and as a subset of the set $X \times Y$, the open cover that covers $X \times Y$ also covers $\{x\} \times Y$. So there exists a finite subcover. This subcover is just a union of open rectangles of the form $U \times V$ (where U is open in X and V is open in Y). Suppose, then, that the open rectangles that make up the finite subcover of the slice $\{x\} \times Y$ is the collection $\{U^x_{\alpha_i} \times V^x_{\beta_i}\}_{i=1}^n$.

We pause to point out that for any x, that $\cup_{i=1}^n V^x_{\beta_i} = Y$ (this was not necessarily obvious in class, but here we address this issue). Let $y \in Y$, I have so show that there exists a p so that $y \in V^x_{\beta_p}$. But then $(x, y) \in \{x\} \times Y$, which is covered by the collection $\{U^x_{\alpha_i} \times V^x_{\beta_i}\}_{i=1}^n$. So there exists a p so that $(x, y) \in U^x_{\alpha_p} \times V^x_{\beta_p}$. This means that $x \in U^x_{\alpha_p}$, and $y \in V^x_{\beta_p}$, and we're done. We'll now continue on with our proof, and use this fact later.

Set $U_x = \cap_{i=1}^n U^x_{\alpha_i}$. This is a finite intersection of open sets, and is thus open. Further, $x \in U_x$. The collection $\{U_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exists a finite subcover, that is, there exists $x_1, \ldots, x_k \in X$ so that $X = \cup_{j=1}^k U_{x_j}$.

Pinky (my associate) claims the following disgusting collection is a finite subcover of rectangles that cover $X \times Y$. The collection is $\{U^{x_j}_{\alpha_i} \times V^{x_j}_{\beta_i}\}_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,k\}}$. We will show that the union of these sets over the i and j listed will be equal to $X \times Y$.

Observe first that $U_x \subseteq U^x_{\alpha_i}$ for any $x \in X$. So

$$\begin{aligned}
\cup_{j=1}^k \cup_{i=1}^n \left[U^{x_j}_{\alpha_i} \times V^{x_j}_{\beta_i} \right] &\supseteq \cup_{j=1}^k \cup_{i=1}^n \left[U_{x_j} \times V^{x_j}_{\beta_i} \right] \\
&= \cup_{j=1}^k \left[U_{x_j} \times \left(\cup_{i=1}^n V^{x_j}_{\beta_i} \right) \right] \\
&= \cup_{j=1}^k \left[U_{x_j} \times Y \right] \\
&= \left[\cup_{j=1}^k U_{x_j} \right] \times Y \\
&= X \times Y \, .
\end{aligned}$$

□

Remark 0.4 I sort of think that Corey's proof in class was more or less this. But I think he did leave out exactly the connection between the rectangles and the open cover we would have been given to start. Also, Corey started to mumble uncontrollably when he spoke of the $V^{x_j}_{\beta_i}$. I think his proof would have been just a little bit better if he included the superscript x stuff as I have in my own proof. See, by leaving those out, it wasn't clear what the correlation was between the finite subcovers of *different* slices. But with the superscripts there, I hope it's a little bit more precise, if not a little more of a headache to read. There will be another handout coming soon regarding the homeomorphisms Corey spoke of in class today. I'm sure someone else will write it, I'm tired of fixing Corey's mistakes! But in any event, ROCK ON!