

Math 610 HW # 3

Hudson, the pinata

April 26, 2007



Hi kiddies! This is Hudson the horse, coming to you from Pinata Island. I'm a part of a ridiculous Saturday morning cartoon that Corey tells me he just can't stand to watch. But, we cleverly put my show between two episodes of Teenage Mutant Ninja Turtles, which Corey sets his Saturday morning alarm for. So he sees some of us each weekend whether he likes it or not! In any event, I managed to stop gazing at myself in my mirror long enough to write this homework assignment for you. Any of the \star questions will also make great presentations. This assignment is due Monday May 7th, But, if enough of you really want to turn it in on Wednesday May 9th, I guess that's okay. But there will be more homework waiting for you! Enjoy! And ROCK ON!!!

1. Regard $S^1 \subseteq \mathbb{C}$, so that it inherits the multiplicative structure in \mathbb{C} . Let $\mathbb{Z}_n = \{\lambda \in S^1 \mid \lambda^n = 1\}$ be the n^{th} roots of unity. Define an equivalence relation \sim as $z_1 \sim z_2$ if and only if $\frac{z_1}{z_2} \in \mathbb{Z}_n$. Let $X_n = S^1 / \sim$. Prove that $X_n \approx S^1$.
2. Use the above problem to show that $\mathbb{R}P^1 \approx S^1$.
3. Prove that $(S^n \times [0, 1]) / (S^n \times \{0\}) \approx D^{n+1}$.
4. \star Let D_+^n be the upper hemisphere in \mathbb{R}^{n+1} , and let D_-^n be the lower hemisphere. Let $f : \partial D_+^n \rightarrow D_-^n$ be the identity map. Show that $D_-^n \cup_f D_+^n \approx S^n$.

5. Let X, Y and Z be spaces. Suppose that Z is a deformation retract of both X and Y . Show that $X \simeq Y$.
6. ★ Let \mathbb{T}^2 be the standard hollow doughnut in \mathbb{R}^3 . Prove that $\mathbb{T}^2 \approx S^1 \times S^1$. Or, you can prove that \mathbb{T}^2 is homeomorphic to the following space:
The space is the unit square $[0, 1] \times [0, 1]$ with the following equivalence relation \sim :
 - (a) $(s, t) \sim (s, t)$.
 - (b) $(0, t) \sim (1, t)$.
 - (c) $(s, 0) \sim (s, 1)$.
7. ★ Let X be a path connected space. Prove $H_0(X) = \mathbb{Z}$.
8. ★ Let $\{X_\alpha\}_{\alpha \in A}$ be distinct path components of X . Prove that $H_n(X) = \bigoplus_{\alpha \in A} H_n(X_\alpha)$.

The following problem is optional:

9. Let X and Y be path connected. Choose a point $x \in X$ and $y \in Y$. Define the “one point union” of X and Y as $X \sqcup Y / \sim$, where $x \sim y$ is the only non-reflexive relation. Call this space $X \vee Y = X \sqcup Y / \sim$.
 - (a) Prove that $X \vee Y \approx (X \times \{y\}) \cup (\{x\} \times Y)$, where the latter is regarded as a subspace of $X \times Y$.
 - (b) The above construction depends on x and y . Let $z \in X$. Prove that the one point union of X with Y where you glue x to y is homotopic to the one point union of X with Y where you glue z to y .
 - (c) Conclude that up to homotopy, $X \vee Y$ does not depend on the point from X or Y that you choose to glue together.