The rest of the proof of the lessons of history

By: My math 212 class

June 13, 2007

Hi, Math 610 kids! This is Corey's math 212 class writing to you to fill in the last bit of the proofs that Corey couldn't quite get out before the end of our class. We've heard so much about you that we had to say hello, and reach out to you with this gesture of good will. Oh, and ROCK ON!

1 The Lessons of History

Theorem 1.1 *The short exact sequence*

$$0 \to C_n(A) \xrightarrow{f} C_n(B) \xrightarrow{g} C_n(C) \to 0$$

gives rise to the natural long exact sequence

$$\cdots H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \cdots$$

Completion of proof. Continuing on from our class notes, we need to show that the choice of equivalence class in $H_n(C)$ doesn't effect $\delta([c])$, completing the proof that δ is well-defined (since we already checked that our choice of $b \in C_n(B)$ didn't effect $\delta([c])$). Then, we will show exactness, and this will complete the entire proof.

Changing our choice of representative in $[c]$ amounts to replacing c by $c + \partial c'$. So we show that this variation doesn't effect the outcome of $\delta([c])$. Since g is onto, there exists a $b, b' \in C_n(B)$ with $g(b) = c, g(b') = c'$. Then $c + \partial c' = g(b) + \partial g(b') = g(b) + g(\partial b') = g(b + \partial b')$. So g maps $b + \partial b'$ to $c + \partial c'$. The next step in computing δ is to move down a degree:

$$b + \partial b' \overset{\partial}{\mapsto} \partial(b + \partial b') = \partial b + \partial^2 b' = \partial b\,.$$

Now it is evident that δ is well defined, since both c and $c + \partial c'$ map to $[a]$ through b by δ.

We prove exactness in 6 steps:

1. $im f_* \subseteq \ker g_*$: We have $g_* f_*[a] = g_*[f(a)] = [g(f(a))] = [0]$, by exactness of the given short exact sequence.

2. $img_* \subseteq \ker\delta$: We have $\delta g_*[b] = \delta[g(b)]$. We would compute $\delta[g(b)]$ by pulling back $g(b)$ to b, then looking at what gets sent to ∂b by f. But $[b] \in H_n(B)$, so $\partial b = 0$, and it's 0 that gets sent to $\partial b = 0$. So $\delta[g(b)] = 0$.

3. $im\delta \subseteq \ker f_*$: Suppose $\delta[c]) = [a]$ via b. Then by definition of δ, $[a]$ is the unique homology class with $f(a) = \partial b$. But $\partial b \in im\partial$, so $[\partial b] = [0]$.

4. $\ker g_* \subseteq imf_*$: Suppose $g_*[b] = 0$. We must find an $[a]$ so that $f_*([a]) = [b]$. Since $g_*[b] = 0$, there exists a c so that $g(b) = \partial c$, and since g is onto, there exists a b' so that $c = g(b')$, and applying ∂ to both sides gives us $\partial c = \partial g(b') = g(\partial b')$. So $g(b) = \partial c = g(\partial b')$, and so $g(b - \partial b') = 0$. By exactness, there exists an a so that $f(a) = b - \partial b'$. We claim that a is the element we are looking for, since $[f(a)] = [b - \partial b'] = [b]$, but we must show that $\partial a = 0$, otherwise it would not represent a homology class. Notice that

$$f(\partial a) = \partial f(a) = \partial(b - \partial b') = \partial b - \partial^2 b' = \partial b = 0.$$

Recall that $[b] \in H_(B)$, and so $\partial b = 0$. But f is injective, so the only element it sends to 0 must be 0 itself, so $\partial a = 0$.

5. $\ker\delta \subseteq img_*$: Suppose $[c] \in H_n(C)$ with $\delta[c] = [0]$ via $\tilde{b}$, then to $\partial a \in im\partial$. We must show that there exists a $[b] \in H_n(B)$ with $g_*[b] = [c]$. We have that $\delta(c) = \partial a$, where by definition of δ, $f(\partial a) = \partial f(a) = \partial\tilde{b}$. So $\partial(\tilde{b} - f(a)) = 0$. Set $b = \tilde{b} - f(a)$. We have just shown that $\partial b = 0$, so $[b] \in H_n(B)$. This is our candidate for the $[b]$ we must find, and all we must do now is show $g_*[b] = [c]$. But $g_*[b] = [g(b)] = [g(\tilde{b} - f(a))] = [g(\tilde{b}) - g(f(a))] = [g(\tilde{b})]$ by our short exact sequence. Since $\delta[c] = [0]$ via $\tilde{b}$, then we have said exactly that $g(\tilde{b}) = c$. And so $g_*[b] = [g(\tilde{b})] = [c]$.

6. $\ker f_* \subseteq im\delta$: Suppose that $f_*[a] = [f(a)] = [0] = [\partial b]$ (that is, $f(a) = \partial b$). We have to show that there exists a $[c] \in H_n(C)$ with $\delta[c] = [a]$. Using b above, we set $c = g(b)$ as our candidate. We have $\partial(g(b)) = g(\partial b) = g(f(a)) = 0$ by the short exact sequence we were given. And in computing $\delta[c]$, since it does not matter which b we choose, we can see that $g(b) = c$, and that the a that gives us $f(a) = \partial b$ is exactly the a we were given at the start. So $\delta[c] = [a]$. □

2 Extra Credit

Wait, what's this? Corey never said anything about extra credit?! What gives? Well, the extra credit is for you, not for your grade. See, there is something in the statement of the lessons of history that Corey never went into detail about. It may have had something to do with the term ending. It's the word "natural". Did anyone ever stop to think that maybe this wasn't something that the Math Gods put in to make it sound better? There's actually a reason why it's there, and we've been implicitly using this ever since we proved that homotopic spaces have isomorphic homology groups in the Mayer-Vietoris sequence.

Naturality means this: suppose that you have a continuous map $h : X \to Y$ with

$h(A) = A', h(B) = B'$, and $h(C) = C'$, and that you have the short exact sequence

$$0 \to C_n(A') \xrightarrow{f'} C_n(B') \xrightarrow{g'} C_n(C') \to 0\,.$$

Then there is an associated long exact sequence in homology (with A', B', C' instead of A, B, C), and that the following diagram is commutative:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\delta} & H_n(A') & \xrightarrow{f'_*} & H_n(B') & \xrightarrow{g'_*} & H_n(C') & \xrightarrow{\delta} & \cdots \\ & \circ & \downarrow h_* & \circ & \downarrow h_* & \circ & \downarrow h_* & \circ & \\ \cdots & \xrightarrow{\delta} & H_n(A) & \xrightarrow{f_*} & H_n(B) & \xrightarrow{g_*} & H_n(C) & \xrightarrow{\delta} & \cdots \end{array}$$

Remember when you all proved that if $X \simeq Y$, then $H_n(X) \cong H_n(Y)$ for all n? And you did this by showing that the homomorphism α_* was an isomorphism, if $\alpha : X \to Y$ and $\beta : Y \to X$ with $\alpha \circ \beta \simeq id_Y$, and $\beta \circ \alpha \simeq id_X$. Well in our computations of the Mayer-Vietoris sequence, we have used the fact that, for instance, if V is contractible, then you may replace IN THE SEQUENCE H_n(a point) for $H_n(V)$. It is this naturality result that allows you to do this, where h_* is the map α_* that I described above. So it is important to include this naturality conclusion as you discuss the lessons of history with your family, your friends, yourself, and your pets.

Other than that, let me just tell you again that I had a great term, and you guys really deserve a lot of credit for sticking with it for so long, and for putting up with me. Good luck to all of you! And, most importantly, don't forget to

Rock on!!!!