Applications of the Fundamental Theorem of Point-Set Topology

Corey's Painful Ankle

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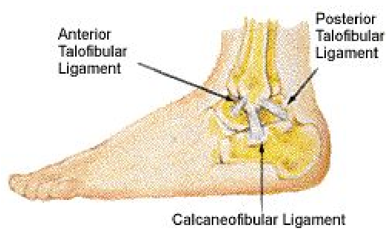

Hello, lovers of topology! This is Corey's left ankle joint coming to you to assist in describing some of the great uses of the Fundamental Theorem of Point-Set Topology. About 4.5 years ago, Corey very badly sprained his left ankle playing basketball–he's never quite been the same, even after 2 surgeries. Sometimes I scream out and make myself heard and keep Corey from doing the things he loves... I figure, he's getting old, may as well get him that much closer to a bridge game with some old ladies! He already plays bridge. Sometimes Corey ingests ibuprofin to keep me quiet, but I always come back! In any event, here is some information that Corey promised he would post. ROCK ON!

For reference, we recall the Fundamental Theorem of Point-Set Topology (FTPST):

Theorem 0.1 *(FTPST) Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

There's also a very important fact about quotient topologies and the induced functions that Corey introduced us to in class. It's important enough to be called a theorem. Normally I like to be pretty precise when I state theorems so the statements are easy to swallow. But there is a lot to be learned from this one, so I've made it just a little more verbose than usual.

Theorem 0.2 *Let X be a space, and $\sim$ be an equivalence relation on X. Let $\pi : X \to X/\sim$ be the canonical projection to the quotient space, endowed with the quotient topology (so a* *set $U \subseteq X/\sim$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X). Let $f : X \to Y$ be a continuous map, so that $x \sim y \Rightarrow f(x) = f(y)$. Then there exists a continuous well-defined induced function $\tilde{f} : X/\sim \to Y$ so that $\tilde{f} \circ \pi = f$ (one might call this a factorization of f through $X/\sim$, or an extension of f).*

Proof. Let $[x]$ be an equivalence class in $X/\sim$. Define $\tilde{f}([x]) = f(x)$. Verify for yourself that this satisfies $\tilde{f}\pi = f$ (it isn't hard). Now by hypothesis we have this function is well defined. That this function is continuous is important, and it comes by way of the definition of the quotient topology. I sort of brushed over this in class, and I think this is a point that deserves a little more discussion.

Let $U \subset Y$ be open. We have to show that $\tilde{f}^{-1}(U)$ is open in $X/\sim$. But sets in $X/\sim$ are open if and only if their preimage under π is open. So we check:

$$\pi^{-1}\left(\tilde{f}^{-1}(U)\right) = (\tilde{f} \circ \pi)^{-1}(U) = f^{-1}(U).$$

But by assumption, f is continuous, and so $f^{-1}(U)$ is open. So $\tilde{f}$ is a continuous function. □

Lemma 0.3 *Use the definitions of D^n and S^n from class. Let*

$$D^n_+ = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} = \left(1 - \sum_{i=1}^{n} x_i^2\right)^{1/2} \right\}.$$

$D^n_+ \subseteq S^n$ is the upper hemisphere of S^n, and endow it with the subspace topology of S^n.

1. *The spaces D^n and D^n_+ are homeomorphic.*
2. *$\partial D^n_+ = \{(x_1, \ldots, x_{n+1}) \in D^n_+ | x_{n+1} = 0\}$.*

Proof. Let $g(x_1, \ldots, x_n) = \left(x_1, \ldots, x_n, (1 - \sum_{i=1}^n x_i^2)^{1/2}\right)$. This is a function $g : D^n \to D^n_+$. It is clearly bijective. Since each $(x_1, \ldots, x_n)$ has norm less than or equal to 1, the map makes sense (no square rooting negative numbers), and is continuous.

The boundary $\partial D^n_+ = \overline{D^n_+} \cap \overline{S^n - D^n_+}$. Now $\overline{D^n_+} = D^n_+$, and $\overline{S^n - D^n_+}$ is the lower hemisphere (including the equator, when the last coordinate $x_{n+1} = 0$). Thus they intersect at $x_{n+1} = 0$. □

Now, we're going to make this easy on ourselves by using spaces that are homeomorphic to the ones listed in class. Or, we could go straight from the definitions and just use the composition of the homeomorphisms described below. I sort of like it this way as it is, in my opinion, more concrete.

We will show that $D^n_+/\partial D^n_+ \approx S^n$, and since the relation "is homeomorphic to" is transitive (it's pretty clearly an equivalence relation, actually), it follows that $D^n/\partial D^n \approx S^n$. We explicitly demonstrate this map when $n = 2$ for clarity. I don't think you'd get much out of it if I used spherical coordinates in arbitrary dimensions. Using spherical coordinates,

$$\begin{aligned} D^2_+ &= \{(r,\theta,\rho) \in \mathbb{R}^3 | r = 1, \theta \in [0, 2\pi], \rho \in [0, \pi/2]\} \\ S^2 &= \{(r,\theta,\rho) \in \mathbb{R}^3 | r = 1, \theta \in [0, 2\pi], \rho \in [0, \pi]\} . \end{aligned}$$

Define the function $F : D^2_+ \to S^2$ in spherical coordinates as $F(r, \theta, \rho) = (r, \theta, 2\rho)$. This map is clearly continuous, and onto. The following Lemma is easy to verify:

Lemma 0.4 *Use the function F just described above.*

1. *Elements on the boundary ∂D^2_+ all get sent to the south pole of S^2.*
2. *The restricted function $\bar{F} : (D^2_+ - \partial D^2_+) \to S^2$ is one to one. (That is, F would be 1-1 if it wasn't for the boundary points all getting sent to the same place.)*

The function F fits into the situation of Theorem 0.2.

Lemma 0.5 *Use the function F described above. Let $\pi : D^2_+ \to D^2_+/\partial D^2_+$ be the projection map that sends a point to the equivalence class represented by that point: $\pi(x) = [x]$. There exists a function $\tilde{F} : D^2_+/\partial D^2_+ \to S^2$ that satisfies (0) $\tilde{F}$ is well-defined, (1) $\tilde{F}$ is continuous, (2) $\tilde{F}$ is bijective, and (3) $F = \tilde{F} \circ \pi$.*

Proof. Define $\tilde{F}([x]) = F(x)$. Should this function be well defined, it automatically satisfies (3) by construction. The equivalence relation mentioned implicitly is the equivalence relation $x \sim x$, and $x \sim y \Leftrightarrow x, y \in \partial D^2_+$. So if $[x] = [y]$, I have to show that $\tilde{F}([x]) = \tilde{F}([y])$. But the only time when $x \neq y$ is when they are both members of ∂D^2_+. And in this case $\tilde{F}([x]) = \tilde{F}([y]) = F(x) = F(y) =$ the south pole, by the previous lemma. So (0) and (3) hold. All of this is just a repeat of the proof of Theorem 0.2, but included in the hope it will increase your understanding of the process. $\tilde{F}$ is continuous by Theorem 0.2.

Now, by the Lemma above, $\tilde{F}$ is one to one, since it agrees with the 1-1 function F everywhere but the boundary, and elements of the boundary are all identified as a single point in $D^2_+/\partial D^2_+$. Also, $\tilde{F}$ is onto since F is onto, establishing the final assertion. □

Theorem 0.6 *The space $D^n/\partial D^n \approx S^n$.*

Proof. The function $\tilde{F}$ we constructed is continuous and bijective. Since D^n is compact (as a closed and bounded subset of $\mathbb{R}^n$), and $D^n/\partial D^n$ is the continuous image of this compact space, $D^n/\partial D^n$ is compact. Furthermore, S^n is clearly Hausdorff. So the FTPST applies, and it follows that $\tilde{F}$ is a homeomorphism. □

Rock on!!!